Semester	III	Corse Title	Engineering Mathematics-III	Course Code	18MAT-31
Teaching Period	50 Hours	L – T – P – TL	2 - 1 - 0 - 3	SEE	3 Hours
CIE	40 Marks	SEE	60Marks	Total	100 Marks
CREDITS - 03					

Course objectives:

- To have an insight into Fourier series, Fourier transforms, Laplace transforms, Difference equations and Z-transforms.
- To develop the proficiency in variational calculus and solving ODE's arising in engineering applications, using numerical methods.

:: Module-1 :(10 Hours)

Laplace Transforms: Definition and Laplace transform of elementary functions. Properties of Laplace transforms(without proof). Laplace transforms of Periodic functions (statement only) and unit-step function – problems.
Inverse Laplace Transforms: Inverse Laplace transform - problems, Convolution theorem to find the inverse Laplace transform (without proof) and problems, solution of linear differential equations using Laplace transforms.
RBTL – L1, L2

:: Module-2 :: (10 Hours)

Fourier series: Periodic functions, Dirichlet's condition. Fourier series of periodic functions period 2π and arbitrary period 2l. Fourier series of even and odd function. Half range Fourier series. Practical harmonic analysis, examples from engineering field.
RBTL –L1, L2

:: Module-3 :: (10 Hours)

Fourier Transforms: Infinite Fourier transforms, Fourier sine and cosine transforms. Inverse Fourier transforms, simple problems.
Difference Equations and Z-Transforms: Difference equations, basic definition, z-transform-definition, Standard z-transforms, Damping and shifting rules, initial value and final value theorems (without proof) and problems, Inverse z-transforms, simple problems.
RBTL –L1, L2

:: Module-4 :: (10 Hours)

Numerical Solutions of Ordinary Differential Equations (ODE's): Numerical solution of ODE's of first order and first degree- Taylor's series method, Modified Euler's method. Runge - Kutta method of fourth order, Milne's and Adam's- Bashforth predictor and corrector method (No derivations of formulae), Problems.
RBTL –L1, L2

:: Module-5 :: (10 Hours)

Numerical Solution of Second Order ODE's: Runge -Kutta method and Milne's predictor and corrector method (No derivations of formulae)-Problems.
Calculus of Variations: Variation of function and functional, variational problems, Euler's equation, Geodesics, hanging chain, problems.
RBTL –L1,L3
L1-Understanding, L2-Remembering, L3-Applying.

Course outcomes:

At the end of the course the student will be able to:

1. CO1: Use Laplace transform and inverse Laplace transform in solving differential/ integral equation arising in network analysis, control systems and other fields of engineering.
2. CO2: Demonstrate Fourier series to study the behavior of periodic functions and their applications in system communications, digital signal processing and field theory.
3. CO3: Make use of Fourier transform and Z-transform to illustrate discrete/continuous function arising in wave and heat propagation, signals and systems.
4. CO4: Solve first and second order ordinary differential equations arising in engineering problems by applying single step and multistep numerical methods.
5. CO5: Determine the extremals of functionals using the calculus of variations and solve problems arising in the dynamics of Rigid bodies and vibration analysis.

Question paper pattern:

- The question paper will have ten full questions carrying equal marks.
- Each full question will be for 20 marks.
- There will be two full questions (with a maximum of three sub- questions) from each module.
- Each full question will have sub- question covering all the topics under a module.

The students will have to answer five full questions, selecting one full question from each module.

Textbooks

1. Advanced Engineering Mathematics, E. Kreyszing, John Wiley & Sons, 10th Edition, 2016.
2. Higher Engineering Mathematics, B.S. Grewal, Khanna Publishers, 44th Edition, 2017.

Reference Books

1. Higher Engineering Mathematics, B.V. Ramana, McGraw-Hill, 11th Edition, 2010.
2. A Text Book of Engineering Mathematics, N. P. Bali and Manish Goyal, Laxmi Publications, 2014.

module - 01

Laplace transforms

Definition:

If $f(t)$ is a real valued function defined for all $t \geq 0$ then the Laplace transform of $f(t)$ denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\, dt$$

provided the integral exists. on integration of the indefinite integral we will be having a function of s and t. when this is evaluated between the limits $t=0$ and $t=\infty$ we will be left with a function of s only and we shall denote it by $\bar{f}(s)$, where s is a parameter, real or complex. thus $L[f(t)] = \bar{f}(s)$

Equivalently we can express this in the form

$$L^{-1}[\bar{f}(s)] = f(t)$$

and is called the inverse laplace transform.

NOTE: $L[c_1 f_1(t) \pm c_2 f_2(t)] = c_1 L[f_1(t)] \pm c_2 L[f_2(t)]$

where c_1 and c_2 are constants

Bernoulli's Rule

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots$$

where u', u'', u''', u^{iv} are successive differentiation

$v_1, v_2, v_3, v_4 \cdots$ are successive integrals of v.

where $V_1 = \int V\,dx$, $V_2 = \int V_1\,dx$

Example

① Find $L[f(t)]$ where $f(t) = \begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$

Solⁿ: $L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$

$$L[f(t)] = \int_0^{4} e^{-st} f(t)\,dt + \int_4^{\infty} e^{-st} f(t)\,dt$$

Using the relevant $f(t)$ in the integrals we have

$$L[f(t)] = \int_0^{4} e^{-st}\, t\,.\,dt + \int_4^{\infty} e^{-st}\,.\,5\,dt$$

$$= \int_0^{4} t\,.\,e^{-st}\,dt + 5\int_4^{\infty} e^{-st}\,dt$$

Using Bernoulli's rule for the first term in RHS we have,

$$L[f(t)] = \left[t\,.\,\frac{e^{-st}}{-s} - (1)\left(\frac{1}{-s}\right)\frac{e^{-st}}{-s}\right]_0^{4} + 5\left[\frac{e^{-st}}{-s}\right]_4^{\infty}$$

$$= -\frac{1}{s}\left[te^{-st} + \frac{e^{-st}}{s}\right]_0^{4} - \frac{5}{s}\left[e^{-st}\right]_4^{\infty}$$

$$= -\frac{1}{s}\left[\left\{4e^{-4s} + \frac{e^{-4s}}{s}\right\} - \left\{0 + \frac{e^{0}}{s}\right\}\right] - \frac{5}{s}\left[e^{-\infty} - e^{-4s}\right]$$

$$= -\frac{1}{s}\left[4e^{-4s} + \frac{e^{-4s}}{s} - \frac{1}{s}\right] - \frac{5}{s}\left[0 - e^{-4s}\right]$$

$$= -\frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} + \frac{5e^{-4s}}{s}$$

$$L[f(t)] = \frac{e^{-4s}}{s} + \frac{1}{s^2}\left(1 - e^{-4s}\right)$$

② Find $L[f(t)]$ if $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

Sol^n: $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f(t)] = \int_0^\pi e^{-st} f(t)\, dt + \int_\pi^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \sin 2t\, dt + \int_\pi^\infty e^{-st}\, 0 . dt$$

w.k.t $\int e^{at} \sin bt\, dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

$$= \left[\frac{e^{-st}}{(-s)^2+2^2}(-s\sin 2t - 2\cos 2t)\right]_{t=0}^{\pi} + 0$$

$$= \frac{-1}{s^2+4}\left[e^{-st}(+s\sin 2t + 2\cos 2t)\right]_{t=0}^{\pi}$$

$$= \frac{-1}{s^2+4}\left[e^{-s\pi}(s\sin 2\pi + 2\cos 2\pi) - e^{0}(s\sin 2(0) + 2\cos 2(0))\right]$$

$\sin 2\pi = 0 = \sin 0$
$\cos 2\pi = 1 = \cos 0$

$$= \frac{-1}{s^2+4}\left[e^{-s\pi}(0+2(1)) - (1)(0+2(1))\right]$$

$$= \frac{-1}{s^2+4}\left[e^{-s\pi}(2) - 2\right]$$

$$= \frac{-2}{s^2+4}\left[e^{-s\pi} - 1\right] = \frac{2}{s^2+4}\left[1 - e^{-s\pi}\right]$$

$\therefore L[f(t)] = \frac{2}{s^2+4}\left[1-e^{-s\pi}\right]$

Laplace transform of elementary functions

① $L(a)$ where a is constant

$$L[f(t)] = L(a) = \int_0^\infty e^{-st}.\, f(t).\, dt$$

$$= \int_0^\infty e^{-st}.\, a\, dt$$

$$= a\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{a}{-s}\left[e^{-st}\right]_0^\infty$$

$$= a/_{-s}\left[e^{-\infty} - e^{0}\right]$$

$$= a/_{-s}\left[0-1\right]$$

$$L(a) = a/s, \text{ where } s>0$$

If $a=1$,
eg: $L(1) = 1/s$

If $a=2$, $L[2] = 2/s$

If $a=100$, $L[100] = \frac{100}{s}$

② $L(e^{at})$

$$L[f(t)] = \int_0^\infty e^{-st}.\, f(t).\, dt$$

$$L[e^{at}] = \int_0^\infty e^{-st}.\, e^{at}.\, dt$$

$$= \int_0^\infty e^{-(s-a)t}.\, dt$$

$$L[e^{at}] = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= \frac{-1}{s-a}\left[e^{-(s-a)t}\right]_0^{\infty}$$

$$= \frac{-1}{s-a}\left[0-1\right]$$

$$L[e^{at}] = \frac{1}{s-a}$$

NOTE: $L[e^{-at}] = \dfrac{1}{s+a}$, $L[e^{-2t}] = \dfrac{1}{s+2}$

$$L[e^{2t}] = \frac{1}{s-2}$$

$$L[e^{3t}] = \frac{1}{s-3}$$

③ $L(\cosh at)$

$$L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right)$$

$$= \frac{1}{2}\left\{L(e^{at}) + L(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{s+\cancel{a}+s-\cancel{a}}{(s-a)(s+a)}\right\}$$

$$= \frac{1}{\cancel{2}}\left\{\frac{\cancel{2}s}{s^2-a^2}\right\}$$

$$= \frac{s}{s^2-a^2}$$

Thus $L(\cosh at) = \dfrac{s}{s^2-a^2}$, where $s>a$

$$L[\cosh at] = \int_0^{\infty} e^{-st}\,\frac{e^{at}+e^{-at}}{2}$$

$$\frac{1}{2}\int_0^{\infty} e^{-st}.e^{at} + e^{-st}$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(s-a)t} + e^{-(s+a)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)}}{s-a} + \frac{e^{-(s+a)}}{s+a}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[0+0+\frac{1}{s-a}+\frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left(\frac{2s}{s^2-a^2}\right)$$

④ $L(\sinh at)$

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2}\left\{L(e^{at}) - L(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{(s+a) - (s-a)}{(s-a)(s+a)}\right\}$$

$$= \frac{1}{2}\left\{\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right\}$$

$$= \frac{1}{\cancel{2}}\left\{\frac{\cancel{2}a}{s^2-a^2}\right\}$$

$$= \frac{a}{s^2-a^2}$$

$$L(\sinh at) = \frac{a}{s^2-a^2}, \quad \text{where } s > a$$

NOTE: $L(\sinh 2t) = \dfrac{2}{s^2-4}$

$$L(\sinh 3t) = \frac{3}{s^2-9}$$

$$L(\sinh t) = \frac{1}{s^2-1}$$

⑤ $L(\cos at)$

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cos at \cdot dt$$

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Using $\int e^{at} \cos bt \, dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$

$$\mathcal{L}(\cos at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{1}{s^2+a^2}\left[e^{-st}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{1}{s^2+a^2}\left[0 - e^{0}(-s\cos(0) + a\sin(0))\right]$$

$$= \frac{1}{s^2+a^2}\left[+s(1) - 0\right]$$

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2} \quad \text{where } s>0$$

⑥ $\mathcal{L}(\sin at)$

$$\mathcal{L}(\sin at) = \int_0^{\infty} e^{-st} \cdot \sin at \, dt$$

using, $\int e^{at} \sin bt \, dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

$$\mathcal{L}(\sin at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\sin at - a\cos at)\right]_{t=0}^{\infty}$$

$$\mathcal{L}(\sin at) = \frac{-1}{s^2+a^2}\left[e^{-st}(s\sin at + a\cos at)\right]_{t=0}^{\infty}$$

$$= \frac{-1}{s^2+a^2}\left[0 - e^{0}(s\sin(0) + a\cos(0))\right]$$

$$= \frac{1}{s^2+a^2}\left[(1)(0+a)\right] \Rightarrow \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$$

(7) $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$, where n is constant

w.k.t $\Gamma(n+1) = n!$, if n is a +ve integer

$L(t^n) = \dfrac{n!}{s^{n+1}}$, If n is a +ve integer.

Table of Laplace transforms

	$f(t)$	$L[f(t)] = \bar{f}(s)$		$f(t)$	$L[f(t)] = \bar{f}(s)$
1.	a	a/s	5	$\sinh at$	$\dfrac{a}{s^2-a^2}$
2.	e^{at}	$\dfrac{1}{s-a}$	6	$\sin at$	$\dfrac{a}{s^2+a^2}$
3.	$\cosh at$	$\dfrac{s}{s^2-a^2}$	7	t^n	$\dfrac{\Gamma(n+1)}{s^{n+1}}$
4.	$\cos at$	$\dfrac{s}{s^2+a^2}$	8	t^n $n=1,2,3\ldots$	$\dfrac{n!}{s^{n+1}}$

problems

(1) Find the Laplace transform of the following functions

(i) $\cosh^2 3t$

Sol^{n}: let, $f(t) = \cosh^2 3t = (\cosh 3t)^2$

$$f(t) = \left[\frac{e^{3t} + e^{-3t}}{2}\right]^2$$

$$\cancel{f(t) = 1/4\,[(e^{3t})^2 + (e^{-3t})^2]}$$

$$\cancel{= 1/4\,[e^{6t} + e^{-6t}}$$

$$f(t) = \frac{1}{2^2}\left[e^{3t} + e^{-3t}\right]^2$$

$\hookrightarrow (a+b)^2$

$$= \frac{1}{4}\left[e^{6t} + e^{-6t} + 2e^{3t}e^{-3t}\right]$$

$$= \frac{1}{4}\left[e^{6t} + e^{-6t} + 2e^{3t-3t}\right]$$

$$= \frac{1}{4}\left[e^{6t} + e^{-6t} + 2e^{0}\right]$$

$$f(t) = \frac{1}{4}\left[e^{6t} + e^{-6t} + 2\right]$$

$$L[f(t)] = \frac{1}{4} L\left[e^{6t} + e^{-6t} + 2\right]$$

$$= \frac{1}{4}\left\{L(e^{6t}) + L(e^{-6t}) + L(2)\right\}$$

$$L[f(t)] = \frac{1}{4}\left\{\frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right\}$$

Do yourself

* $\sinh^2 2t$

(2) $e^{-2t}\sinh 4t$

Sol^{n}: Let, $f(t) = e^{-2t}\sinh 4t$

w.k.t $\sinh t = \frac{e^{t} - e^{-t}}{2}$

$$\sinh 4t = \frac{e^{4t} - e^{-4t}}{2}$$

$$f(t) = e^{-2t}\sinh 4t = e^{-2t}\left[\frac{e^{4t} - e^{-4t}}{2}\right]$$

$$= \frac{1}{2}\left[e^{-2t}\cdot e^{4t} - e^{-2t}\cdot e^{-4t}\right]$$

$$= \frac{1}{2}\left[e^{2t} - e^{-6t}\right]$$

$$f(t) = \frac{1}{2}\left[e^{2t} - e^{-6t}\right]$$

$$L[f(t)] = \frac{1}{2}\left[L(e^{2t}) - L(e^{-6t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{S-2} - \frac{1}{S+6}\right]$$

$$= \frac{1}{2}\left[\frac{(S+6)-(S-2)}{(S-2)(S+6)}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{S}+6-\cancel{S}+2}{(S-2)(S+6)}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{8}\,4}{(S-2)(S+6)}\right]$$

$$L[f(t)] = \boxed{\frac{4}{(S-2)(S+6)}}$$

③ $\sin 5t \cdot \cos 2t$

Solⁿ: Let $f(t) = \sin 5t \cos 2t$

W.K.T $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$$f(t) = \frac{1}{2}\left[\sin(5t+2t) + \sin(5t-2t)\right]$$

$$f(t) = \frac{1}{2}\left[\sin 7t + \sin 3t\right]$$

$$L[f(t)] = \frac{1}{2}\left[L(\sin 7t) + L(\sin 3t)\right]$$

W.K.T $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\frac{7}{s^2+7^2} + \frac{3}{s^2+3^2}\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

(4) $\cos t \cdot \cos 2t \cdot \cos 3t$

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Sol^{n}: Let $f(t) = \cos t \cos 2t \cos 3t$

W.K.T $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$\cos t \cos 2t = \frac{1}{2}[\cos(t+2t) + \cos(t-2t)]$$

$$= \frac{1}{2}[\cos 3t + \cos(t)]$$

$$= \frac{1}{2}[\cos 3t + \cos t]$$

$$\therefore \cos t \cos 2t \cos 3t = \frac{1}{2}[\cos 3t + \cos t]\cos 3t$$

$$= \frac{1}{2}[\cos 3t \cos 3t + \cos t \cos 3t]$$

$$= \frac{1}{2}\left[\frac{1}{2}[\cos(3t+3t) + \cos(3t-3t)] + \frac{1}{2}[\cos(t+3t) + \cos(t-3t)]\right]$$

$$= \frac{1}{2} \times \frac{1}{2}\left[(\cos 6t + \cos 0) + (\cos 4t + \cos(-2t))\right]$$

$$f(t) = \frac{1}{4}\left[\cos 6t + 1 + \cos 4t + \cos 2t\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{4}\left[\mathcal{L}[\cos 6t] + \mathcal{L}[1] + \mathcal{L}[\cos 4t] + \mathcal{L}[\cos 2t]\right]$$

W.K.T $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$, $\mathcal{L}[a] = \frac{a}{s}$

$$= \frac{1}{4}\left[\frac{s}{s^2+6^2} + \frac{1}{s} + \frac{s}{s^2+4^2} + \frac{s}{s^2+2^2}\right]$$

$$L[f(t)] = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{1}{s} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$$

(5) $\sin^2(2t+1)$

Sol^n: Let $f(t) = \sin^2(2t+1)$

w.k.t $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$f(t) = \sin^2(2t+1)$$

$$= \frac{1-\cos 2(2t+1)}{2}$$

$$= \frac{1}{2}\left[1-\cos(4t+2)\right]$$

$$= \frac{1}{2}\left[1-\{\cos 4t \cos 2 - \sin 4t \sin 2\}\right]$$

$$f(t) = \frac{1}{2}\left[1 - \cos 4t \cos 2 + \sin 4t \sin 2\right]$$

$$L[f(t)] = \frac{1}{2}\left[L(1) - L(\cos 4t)\cos 2 + \sin 2\, L(\sin 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\cos 2 + \sin 2 \cdot \frac{4}{s^2+4^2}\right]$$

$$L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s\cdot\cos 2}{s^2+16} + \frac{4\sin 2}{s^2+16}\right]$$

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(6) $(3t+4)^3 + 5^t$

Sol: Let $f(t) = (3t+4)^3 + 5^t$

W.K.T $(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$(3t+4)^3 = (3t)^3 + 4^3 + 3(3t)^2(4) + 3(3t)(4)^2 + e^{\log 5 \cdot t}$

$= 27t^3 + 64 + 108t^2 + 144t + e^{\log 5 \cdot t}$

$L[(3t+4)^3] = 27L[t^3] + L[64] + 108L[t^2] + 144L[t]$
$+ L[e^{\log 5 \cdot t}]$

$= 27 \cdot \frac{3!}{s^4} + \frac{64}{s} + 108 \cdot \frac{2!}{s^3} + 144 \cdot \frac{1}{s^2}$
$+ \frac{1}{s-\log 5}$

$= \frac{162}{s^4} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{64}{s} + \frac{1}{s-\log 5}$

(7) $f(t) = 3\sqrt{t} + \frac{4}{\sqrt{t}}$

$f(t) = 3t^{1/2} + 4/t^{1/2}$

$f(t) = 3t^{1/2} + 4t^{-1/2}$

W.K.T $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

$L[f(t)] = L[3t^{1/2}] + L[4t^{-1/2}]$

$= 3L[t^{1/2}] + 4L[t^{-1/2}]$

$= 3\left[\frac{\Gamma(1/2+1)}{s^{1/2+1}}\right] + 4\left[\frac{\Gamma(-1/2+1)}{s^{-1/2+1}}\right]$

$L[f(t)] = 3\frac{\Gamma(3/2)}{s^{3/2}} + 4\frac{\Gamma(1/2)}{s^{1/2}}$

$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$

$L[f(t)] = 3\frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} + 4\frac{\sqrt{\pi}}{s^{1/2}}$

$$= 3\frac{\sqrt{\pi}}{2S^{3/2}} + \frac{4\sqrt{\pi}}{S^{1/2}}$$

$$= \frac{3}{2}\frac{\sqrt{\pi}}{S^{1/2}\cdot S} + \frac{4\sqrt{\pi}}{S^{1/2}}$$

$$= \frac{\sqrt{\pi}}{S^{1/2}}\left[\frac{3}{2S} + 4\right]$$

$$L[f(t)] = \sqrt{\frac{\pi}{S}}\left[\frac{3}{2S} + 4\right]$$

Properties of Laplace transforms

① If $L[f(t)] = \bar{f}(S)$ then $L[e^{at} f(t)] = \bar{f}(S-a)$

② If $L[f(t)] = \bar{f}(S)$, then

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(S)]$ where n is a +ve integer. In particular $L[tf(t)] = -\frac{d}{ds}[\bar{f}(S)]$

$L[t^2 f(t)] = \frac{d^2}{ds^2}[\bar{f}(S)]$

③ If $L[f(t)] = \bar{f}(S)$, then $L\left[\frac{f(t)}{t}\right] = \int_S^{\infty} \bar{f}(S)\, ds$

Problems

* Find the Laplace transform of the following functions

① $e^{-2t}(2\cos 5t - \sin 5t)$

Soln: Let $f(t) = 2\cos 5t - \sin 5t$

$$L[f(t)] = 2L[\cos 5t] - L[\sin 5t]$$

$$= 2 \cdot \frac{S}{S^2+5^2} - \frac{5}{S^2+5^2}$$

$$L[f(t)] = \frac{2S-5}{S^2+5^2}$$

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$$\mathcal{L}[e^{-2t} f(t)] = \left\{\frac{2s-5}{s^2+25}\right\}_{s \to s+2}$$

$$= \frac{2(s+2)-5}{(s+2)^2+25}$$

$$= \frac{2s+4-5}{s^2+2^2+4s+25}$$

$$\mathcal{L}[e^{-2t} f(t)] = \frac{2s-1}{s^2+4s+29}$$

$$\therefore \mathcal{L}[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2s-1}{s^2+4s+29}$$

② $e^{-t}\cos^2 3t$

Sol^n: W.K.T $\cos^2 t = \frac{1+\cos 2t}{2}$

$$f(t) = \cos^2 3t = \frac{1+\cos 6t}{2}$$

$$\mathcal{L}[f(t)] = \mathcal{L}\left[\frac{1+\cos 6t}{2}\right]$$

$$= \frac{1}{2}\mathcal{L}[1+\cos 6t]$$

$$= \frac{1}{2}\left\{\mathcal{L}[1] + \mathcal{L}[\cos 6t]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+6^2}\right\}$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

$$\mathcal{L}[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]_{s \to s+1}$$

$$\mathcal{L}[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2+36}\right]$$

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(3) $(1+3te^{2t})^2$

Sol^n: Let $f(t) = (1+3te^{2t})^2$

use $(a+b)^2 = a^2+b^2+2ab$

$(1+3te^{2t})^2 = 1+(3te^{2t})^2 + 2(1)(3te^{2t})$

$(1+3te^{2t})^2 = 1+9t^2e^{4t} + 6te^{2t}$

$\mathcal{L}[f(t)] = \mathcal{L}[1+9t^2e^{4t}+6te^{2t}]$

$= \mathcal{L}[1] + 9\mathcal{L}[e^{4t}\cdot t^2] + 6\mathcal{L}[te^{2t}]$

$= \frac{1}{s} + 9\mathcal{L}[t^2]_{s\to s-4} + 6\mathcal{L}[t]_{s\to s-2}$

But $\mathcal{L}(t) = 1/s^2$ $\quad \mathcal{L}(t^2) = 2/s^3$

use $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$\mathcal{L}[(1+3te^{2t})^2] = 1/s + 9\cdot\frac{2}{(s-4)^3} + \frac{6}{(s-2)^2}$

$\mathcal{L}[(1+3te^{2t})^2] = 1/s + \frac{18}{(s-4)^3} + \frac{6}{(s-2)^2}$

(4) Sinhat Sinat

Sol^n: $f(t) = \sinh at \sin at$

$f(t) = \frac{e^{at}-e^{-at}}{2}\cdot \sin at$

$f(t) = 1/2\,(e^{at}\sin at - e^{-at}\sin at)$

$\mathcal{L}[f(t)] = 1/2\left\{\mathcal{L}[\sin at]_{s\to s-a} - \mathcal{L}[\sin at]_{s\to s+a}\right\}$

But $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$

$= 1/2\left\{\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right\}$

$= 1/2\left\{\frac{a}{s^2+a^2-2as+a^2} - \frac{a}{s^2+a^2+2as+a^2}\right\}$

$$= \frac{a}{2}\left\{\frac{1}{s^2+2a^2-2as} + \frac{1}{s^2+2a^2+2as}\right\}$$

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$$= \frac{a}{2}\left\{\frac{(s^2+2a^2+2as)-(s^2+2a^2-2as)}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right\}$$

$(s^2+2a^2-2as)(s^2+2a^2+2as)$

$s^4+2s^2a^2+\cancel{2as^3}+$

$2s^2a^2+4a^4+\cancel{4a^3s}$

$\cancel{-2as^3}-\cancel{4a^3s}-$

$\cancel{4a^2s^2}$

s^4+4a^4

$$= \frac{a}{2}\left\{\frac{2as+2as}{s^4+4a^4}\right\}$$

$$= \frac{a}{2}\left\{\frac{4as}{s^4+4a^4}\right\}$$

(8) $\sin^3 2t \cosh t$

Sol^n: Let $f(t) = \sin^3 2t$

$= \frac{1}{4}\{3\sin 2t + \sin$

W.K.T $\sin 3t = 3\sin t - 4\sin^3 t$

$4\sin^3 t = 3\sin t - \sin 3t$

$\sin^3 t = \frac{1}{4}[3\sin t - \sin 3t]$

$\sin^3 2t = \frac{1}{4}[3\sin 2t - \sin 3(2t)]$

$\sin^3 2t = \frac{1}{4}[3\sin 2t - \sin 6t]$

$$L[\sin^3 2t] = \frac{1}{4}\{L[3\sin 2t] - L[\sin 6t]\}$$

$$= \frac{1}{4}\{3L[\sin 2t] - L[\sin 6t]\}$$

$$= \frac{1}{4}\left\{3\cdot\frac{2}{s^2+4} - \frac{6}{s^2+36}\right\}$$

$$= \frac{1}{4}\left\{\frac{6}{s^2+4} - \frac{6}{s^2+36}\right\}$$

$$= \frac{1}{4}\left\{\frac{6(s^2+36)-6(s^2+4)}{(s^2+4)(s^2+36)}\right\}$$

$$= \frac{1}{4}\left\{\frac{6s^2+216-6s^2-24}{(s^2+4)(s^2+36)}\right\}$$

$$= \frac{1}{4}\left\{\frac{192}{(s^2+4)(s^2+36)}\right\}$$

$$= \frac{48}{(s^2+4)(s^2+36)}$$

Now, $\mathcal{L}(\cosh t \sin^3 2t) = \mathcal{L}\left[\frac{e^t+e^{-t}}{2} \cdot \sin^3 2t\right]$

$$= \frac{1}{2}\left\{\mathcal{L}\left[e^t \cdot \sin^3 2t\right] + \mathcal{L}\left[e^{-t} \cdot \sin^3 2t\right]\right\}$$

$$= \frac{1}{2}\left\{\mathcal{L}\left[\sin^3 2t\right]_{s \to s-1} + \mathcal{L}\left[\sin^3 2t\right]_{s \to s+1}\right\}$$

$$= \frac{1}{2}\left[\frac{48}{((s-1)^2+4)((s-1)^2+36)} + \frac{48}{((s+1)^2+4)((s+1)^2+36)}\right]$$

$$= \frac{48}{2}\left[\frac{1}{((s-1)^2+4)((s-1)^2+36)} + \frac{1}{((s+1)^2+4)((s+1)^2+36)}\right]$$

$$= 24\left[\frac{1}{(s^2-2s+5)(s^2-2s+37)} + \frac{1}{(s^2+2s+5)(s^2+2s+37)}\right]$$

Find the Laplace transform of the following functions

① $t\cos at$

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Soln: Let $f(t) = \cos at$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+a^2}$$

Now $\mathcal{L}[t f(t)] = (-1)\frac{d}{ds}[\bar{f}(s)]$

$$\mathcal{L}[t\cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left\{\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right\}$$

$$= -\left\{\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right\}$$

$$= -\left\{\frac{a^2-s^2}{(s^2+a^2)^2}\right\}$$

$$\mathcal{L}[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

② $t^2 \sin at$

Solⁿ: let $f(t) = \sin at$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin at]$$

$$= \frac{a}{s^2+a^2}$$

Now $\mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[\bar{f}(s)]$

$$= \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right\}$$

$$= \frac{d}{ds}\left\{\frac{(s^2+a^2)(0) - a(2s)}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{-2as}{(s^2+a^2)^2}\right\}$$

$$= \left\{\frac{(s^2+a^2)^2(-2a) - (-)2as\,[2(s^2+a^2)(2s)]}{((s^2+a^2)^2)^2}\right\}$$

$$= \frac{\cancel{(s^2+a^2)}\,2a\left\{-(s^2+a^2) + 4s^2\right\}}{\cancel{(s^2+a^2)^4}}$$

$$L[t^2 \sin at] = \frac{2a\left\{-(s^2+a^2) + 4s^2\right\}}{(s^2+a^2)^3}$$

$$= \frac{2a\left\{-s^2 - a^2 + 4s^2\right\}}{(s^2+a^2)^3}$$

$$= \frac{2a\left\{3s^2 - a^2\right\}}{(s^2+a^2)^3}$$

③ $t^3 \sin t$

Sol^n: Let $f(t) = \sin t$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] \quad \because \mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[f(t)] = \frac{1}{s^2+1} \longrightarrow \text{property ②}$$

Now, $$\mathcal{L}[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[\bar{f}(s)]$$

$$= (-1)^3 \frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{(s^2+1)(0)-(1)(2s)}{(s^2+1)^2}\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= +\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{2s}{(s^2+1)^2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)^2(2) - 2s\left(2(s^2+1)(2s)\right)}{\left((s^2+1)^2\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^4+1+2s^2)(2) - 8s^2(s^2+1)}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{2s^4+2+4s^2-8s^4-8s^2}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{-6s^4-4s^2+2}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{\cancel{(s^2+1)}\left\{2(s^2+1)-8s^2\right\}}{\cancel{(s^2+1)^4}}\right]$$

$$= \frac{d}{ds}\left[\frac{2s^2+2-8s^2}{(s^2+1)^3}\right]$$

$$= \frac{d}{ds}\left[\frac{2-6s^2}{(s^2+1)^3}\right]$$

$$= 2\frac{d}{ds}\left[\frac{1-3s^2}{(s^2+1)^3}\right]$$

$$= 2\left\{\frac{(s^2+1)^3(0-6s)-(1-3s^2)(3(s^2+1)^2(2s))}{((s^2+1)^3)^2}\right\}$$

$$= 2\left\{\frac{(s^2+1)^3(-6s)-6s(1-3s^2)(s^2+1)^2}{(s^2+1)^6}\right\}$$

$$= 2\left\{\frac{\cancel{(s^2+1)^2}\left\{-6s(s^2+1)-6s(1-3s^2)\right\}}{\cancel{(s^2+1)^6}}\right\}$$

$$= 2\left\{\frac{-6s\left\{s^2+1+1-3s^2\right\}}{(s^2+1)^4}\right\}$$

$$= \frac{-12s(2-2s^2)}{(s^2+1)^4}$$

$$= \frac{-24s(1-s^2)}{(s^2+1)^4} \Rightarrow \mathcal{L}\left[t^3 f(t)\right] = \frac{24s(s^2-1)}{(s^2+1)^4}$$

④ $t^3 \cosh t$

Solⁿ: Let $f(t) = t^3 \cosh t$

NOTE: (Just for understanding) Here we should not prefer to work the problem $\|^{ly}$ to previous, since we have $\cosh t$ which can be converted to $\frac{e^t + e^{-t}}{2}$ so that it will be highly convenient to apply the shifting property.

$$f(t) = t^3\left(\frac{e^t + e^{-t}}{2}\right)$$

$$f(t) = \tfrac{1}{2}\left\{t^3 e^t + t^3 e^{-t}\right\}$$

$$f(t) = \tfrac{1}{2}\left\{e^t \cdot t^3 + e^{-t} \cdot t^3\right\} \rightarrow \text{use property ①}$$

$$\mathcal{L}[f(t)] = \tfrac{1}{2}\left\{\mathcal{L}[t^3]_{s \to s-1} + \mathcal{L}[t^3]_{s \to s+1}\right\}$$

w.k.t $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ $\therefore \mathcal{L}[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

$$\mathcal{L}[f(t)] = \tfrac{1}{2}\left\{\frac{6}{(s-1)^4} + \frac{6}{(s+1)^4}\right\}$$

$$= \tfrac{6}{2}\left\{\frac{1}{(s-1)^4} + \frac{1}{(s+1)^4}\right\}$$

$$\mathcal{L}[f(t)] = 3\left\{\frac{1}{(s-1)^4} + \frac{1}{(s+1)^4}\right\}$$

⑤ $t^5 e^{4t} \cosh 3t$

Solⁿ: $f(t) = t^5 e^{4t} \cosh 3t$

$$f(t) = t^5 e^{4t}\left\{\frac{e^{3t} + e^{-3t}}{2}\right\}$$

$$f(t) = \frac{e^{7t} \cdot t^5 + e^{t} \cdot t^5}{2}$$

$$f(t) = \tfrac{1}{2}\left\{e^{7t} \cdot t^5 + e^{t} \cdot t^5\right\}$$

$$\mathcal{L}[f(t)] = \tfrac{1}{2}\left\{\mathcal{L}[t^5]_{s \to s-7} + \mathcal{L}[t^5]_{s \to s-1}\right\} \text{ (by property ①)}$$

W.K.T $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$$\mathcal{L}[t^5] = \frac{5!}{s^{5+1}}$$

$$\mathcal{L}[t^5] = \frac{120}{s^6}$$

$$\mathcal{L}[t^5 e^{4t}\cosh 3t] = \frac{1}{2}\left\{\frac{120}{(s-7)^6} + \frac{120}{(s-1)^6}\right\}$$

$$= \frac{120}{2}\left\{\frac{1}{(s-7)^6} + \frac{1}{(s-1)^6}\right\}$$

$$\mathcal{L}[t^5 e^{4t}\cosh 3t] = 60\left\{\frac{1}{(s-7)^6} + \frac{1}{(s-1)^6}\right\}$$

(6) $t e^{-2t} \sin 4t$

Soln: let $f(t) = t e^{-2t} \sin 4t$

$$\mathcal{L}[\sin 4t] = \frac{4}{s^2+16}$$

$$\therefore \mathcal{L}[e^{-2t}\sin 4t] = \mathcal{L}\left[\frac{4}{s^2+16}\right]_{s \to s+2}$$

$$= \frac{4}{(s+2)^2+16}$$

$$\mathcal{L}[e^{-2t}\sin 4t] = \frac{4}{s^2+4+4s+16}$$

$$\mathcal{L}[e^{-2t}\sin 4t] = \frac{4}{s^2+4s+20}$$

$$\mathcal{L}[t e^{-2t}\sin 4t] = -\frac{d}{ds}\left[\frac{4}{s^2+4s+20}\right]$$

$$= -\left\{\frac{(s^2+4s+20)(0) - 4(2s+4)}{(s^2+4s+20)^2}\right\}$$

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$$= \frac{4(2s+4)}{(s^2+4s+20)^2}$$

$$= \frac{8(s+2)}{(s^2+4s+20)^2}$$

(7) S.T $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$

Sol^n: W.K.T $\int_0^\infty e^{-st} f(t)\, dt = \mathcal{L}[f(t)]$

$$\int_0^\infty e^{-st} t^3 \sin t \, dt = \mathcal{L}[t^3 \sin t] \quad \text{—— ①}$$

Now we have to find $\mathcal{L}[t^3 \sin t]$, $f(t) = \sin t$

$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] = \frac{1}{s^2+1}$

$$\mathcal{L}[t^3 \sin t] = -\frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right] \quad \{\text{property ②}\}$$

$$= -\frac{d^2}{ds^2}\left\{\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\}\right\}$$

$$= -\frac{d^2}{ds^2}\left\{\frac{(s^2+1)(0) - (1)(2s)}{(s^2+1)^2}\right\}$$

$$= -\frac{d^2}{ds^2}\left\{\frac{-2s}{(s^2+1)^2}\right\} = \frac{d^2}{ds^2}\left\{\frac{2s}{(s^2+1)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{d}{ds}\left\{\frac{2s}{(s^2+1)^2}\right\}\right\}$$

$$= \frac{d}{ds}\left\{\frac{(s^2+1)^2(2) - 2s(2(s^2+1)(2s))}{(s^2+1)^4}\right\}$$

$$= \frac{d}{ds}\left\{\frac{\cancel{(s^2+1)}(2)\{(s^2+1) - 4s^2\}}{(s^2+1)^{\cancel{4}}}\right\}$$

$$= 2\frac{d}{ds}\left\{\frac{s^2+1-4s^2}{(s^2+1)^3}\right\}$$

$$= 2\frac{d}{ds}\left\{\frac{1-3s^2}{(s^2+1)^3}\right\}$$

$$= 2\left\{\frac{(s^2+1)^3(-6s) - (1-3s^2)(3(s^2+1)^2(2s))}{((s^2+1)^3)^2}\right\}$$

$$= 2\left\{\cancel{(s^2+1)^2}(-6s)\left\{\frac{(s^2+1)+(1-3s^2)}{\cancel{(s^2+1)}^6}\right\}\right\}$$

$$= -12s\left\{\frac{-2s^2+2}{(s^2+1)^4}\right\}$$

$$= -24s\left\{\frac{-s^2+1}{(s^2+1)^4}\right\}$$

$$= 24s\left\{\frac{s^2-1}{(s^2+1)^4}\right\}$$

Thus ① becomes

$$\int_0^\infty e^{-st}t^3\sin t\,dt = 24s\left\{\frac{s^2-1}{(s^2+1)^4}\right\}$$

put $s=1$

$$\int_0^\infty e^{-t}t^3\sin t\,dt = 24\left\{\frac{1-1}{(2)^4}\right\}$$

$$= 24(0)$$

$$\int_0^\infty e^{-t}t^3\sin t\,dt = 0$$

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(8) S.T $\int_0^\infty t e^{-2t} \sin 4t \, dt = 1/25$

Soln: W.K.T $\int_0^\infty e^{-st} f(t)\, dt = \mathcal{L}[f(t)]$

$$\int_0^\infty e^{-st} t \sin 4t \, dt = \mathcal{L}[t \sin 4t] \rightarrow (1)$$

$\mathcal{L}[t\sin 4t]$ → {property (2)}

$f(t) = \sin 4t, \quad \mathcal{L}[f(t)] = \mathcal{L}[\sin 4t] = \frac{4}{s^2+16}$

$$\mathcal{L}[t\sin 4t] = -\frac{d}{ds}\left[\frac{4}{s^2+16}\right]$$

$$= -\left[\frac{(s^2+16)(0) - 4(2s)}{(s^2+16)^2}\right]$$

$$\mathcal{L}[t\sin 4t] = \frac{8s}{(s^2+16)^2}$$

$$(1) \Rightarrow \int_0^\infty e^{-st} t \sin 4t = \frac{8s}{(s^2+16)^2}$$

Thus, put $s = 2$

$$\int_0^\infty e^{-2t} t \sin 4t = \frac{8 \times 2}{(2^2+16)^2}$$

$$\int_0^\infty e^{-2t} t \sin 4t = 16/400$$

$$\int_0^\infty e^{-2t} t \sin 4t = 1/25$$

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* Find the value of $\int_0^\infty t e^{-3t} \cos 2t \, dt$ using Laplace transform.

Ans: $\int_0^\infty e^{-3t} \cdot t \cos 2t \, dt = \frac{5}{169}$

Find the Laplace transform of the following functions

① $\frac{1-e^{-at}}{t}$

Solⁿ: Use property 3. $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$

Let $f(t) = 1-e^{-at}$

$$L[f(t)] = L[1-e^{-at}]$$

$$= L[1] - L[e^{-at}]$$

$$= \frac{1}{s} - \frac{1}{s+a}$$

$$\therefore L[f(t)] = \frac{1}{s} - \frac{1}{s+a}$$

$$L[f(t)] = \bar{f}(s)$$

we have $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+a}\right) ds$$

$$= \left[\log s - \log(s+a)\right]_s^\infty$$

$\hookrightarrow \{\log m - \log n = \log(m/n) \text{ form}\}$

$$= \left[\log\left(\frac{s}{s+a}\right)\right]_s^{\infty}$$

$$= \left[\lim_{s\to\infty} \log\left(\frac{s}{s+a}\right)\right] - \log\left(\frac{s}{s+a}\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{s}{s(1+a/s)}\right) - \log\left(\frac{s}{s+a}\right)$$

$$= \log\left(\frac{1}{1+0}\right) - \log\left(\frac{s}{s+a}\right)$$

$$= \log(1) - \log\left(s/s+a\right)$$

$$= 0 - \log\left(s/s+a\right)$$

$$= \log\left(\frac{s+a}{s}\right)$$

$\log a^{-m} = \log \frac{1}{a^m}$

$$= -\log\left(\frac{s}{s+a}\right) = \log\left(\frac{s}{s+a}\right)^{-1} = \log\left(\frac{s+a}{s}\right)$$

② $\dfrac{\sin^2 t}{t}$

Solⁿ: Let $f(t) = \sin^2 t$

$$f(t) = \frac{1-\cos 2t}{2}$$

$$\therefore \mathcal{L}[f(t)] = \mathcal{L}\left[\frac{1-\cos 2t}{2}\right]$$

$$= \frac{1}{2}\{\mathcal{L}[1] - \mathcal{L}[\cos 2t]\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}$$

$$\mathcal{L}[f(t)] = \bar{f}(s)$$

hence $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\, ds$

$$= \frac{1}{2}\int_s^{\infty}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\} ds$$

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\left[\left\{\log s - \frac{1}{2}\log(s^2+4)\right\}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log(s^2+4)^{1/2}\right]_s^\infty$$

w.k.T $m \log n = \log n^m$

$$= \frac{1}{2}\left[\log s - \log(s^2+4)^{1/2}\right]_s^\infty$$

$\log m - \log n = \log(m/n)$

$$= \frac{1}{2}\left[\log\left(\frac{s}{(s^2+4)^{1/2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2(1+4/s^2)}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{\cancel{s}}{\cancel{s}\sqrt{1+4/s^2}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\log\left[\frac{1}{\sqrt{1+0}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\log(1) - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= 0 - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\left[\frac{\sqrt{s^2+4}}{s}\right]$$

③ $\frac{2\sin t \sin 5t}{t}$

Sol^n: let $f(t) = 2\sin t \sin 5t$

w.k.T $\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

$\sin t \sin 5t = -\frac{1}{2}\left[\cos 6t - \cos(-4t)\right]$

$= -\frac{1}{2}\left[\cos 6t - \cos 4t\right]$

$\sin t \sin 5t = \frac{1}{2}\left[\cos 4t - \cos 6t\right]$

Now, $f(t) = \cancel{2} \cdot \frac{1}{\cancel{2}}\left[\cos 4t - \cos 6t\right]$

$f(t) = \cos 4t - \cos 6t$

$\mathcal{L}[f(t)] = \mathcal{L}[\cos 4t] - \mathcal{L}[\cos 6t]$

$= \frac{s}{s^2+16} - \frac{s}{s^2+36}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$

$= \int_s^\infty \left\{\frac{s}{s^2+16} - \frac{s}{s^2+36}\right\} ds$

$= \left[\frac{1}{2}\log(s^2+16) - \frac{1}{2}\log(s^2+36)\right]_s^\infty$

$= \frac{1}{2}\left[\log(s^2+16) - \log(s^2+36)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]_s^\infty$

$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{\cancel{s^2}(1+16/s^2)}{\cancel{s^2}(1+36/s^2)}\right] - \frac{1}{2}\log\left[\frac{s^2+16}{s^2+36}\right]$

$$= \frac{1}{2}\left\{\cancel{\log(1)}^{0} - \log\left(\frac{s^2+16}{s^2+36}\right)\right\}$$

$$= \frac{1}{2}\left\{-\log\left(\frac{s^2+16}{s^2+36}\right)\right\}$$

$$= \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)$$

$$\Rightarrow m\log n = \log n^m$$

$$= \log\left(\frac{s^2+36}{s^2+16}\right)^{1/2}$$

$$L\left[\frac{2\sin t \sin 5t}{t}\right] = \log\sqrt{\frac{s^2+36}{s^2+16}}$$

④ $\frac{\sin at}{t}$ Dec 2018

Solⁿ: Let $f(t) = \sin at$

$$L[f(t)] = L[\sin at]$$

$$L[f(t)] = \frac{a}{s^2+a^2}$$

$$L[f(t)] = \bar{f}(s)$$

hence $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$L\left[\frac{\sin at}{t}\right] = a \times \frac{1}{a}\left[\tan^{-1}(s/a)\right]_s^\infty$$

$$\frac{d}{dx}\left(\tan^{-1} x/a\right)$$

$$= \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\tan^{-1} x/a\right) = \frac{1}{1+\frac{x^2}{a^2}} \times \frac{1}{a}$$

$$= \frac{a}{a^2+x^2}$$

$= \frac{\infty}{a}\left[\tan^{-1}(s/a)\right]_s^{\infty}$

$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$

$= \pi/2 - \tan^{-1}(s/a)$

W.K.T

$\cot^{-1} x = \pi/2 - \tan^{-1} x$

$\tan \pi/2 = \infty$

$\therefore \pi/2 = \tan^{-1}\infty$

$\mathcal{L}\left[\frac{\sin at}{t}\right] = \cot^{-1}(s/a)$

⑥ $\frac{\cos at - \cos bt}{t}$ (ACU Jan 2020)

Solⁿ: Let $f(t) = \cos at - \cos bt$

$\mathcal{L}[f(t)] = \mathcal{L}[\cos at - \cos bt]$

$= \mathcal{L}[\cos at] - \mathcal{L}[\cos bt]$

$\mathcal{L}[f(t)] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\,ds$

$= \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^{\infty}$

$= \frac{1}{2}\left[\log\left\{\frac{s^2+a^2}{s^2+b^2}\right\}\right]_s^{\infty}$

$= \frac{1}{2}\left\{\lim_{s\to\infty} \log\left(\frac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

$= \frac{1}{2}\left\{\cancel{\log(1)}^{\,0} - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

$= \frac{1}{2}\left\{-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

$$= \frac{1}{2} \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

NOTE: * $m \log n = \log n^m$

eg: $-\log\left(\frac{s}{s+a}\right) = \log\left(\frac{s}{s+a}\right)^{-1}$

$$= \log\left(\frac{1}{\frac{s}{s+a}}\right)$$

$$-\log\left(\frac{s}{s+a}\right) = \log\left(\frac{s+a}{s}\right)$$

* $\log(mn) = \log m + \log n$
* $\log\left(\frac{m}{n}\right) = \log m - \log n$

Doyourself

* Find the Laplace transform of $\frac{\sinh t}{t}$

Ans: $L\left[\frac{\sinh t}{t}\right] = \log\sqrt{\frac{s+1}{s-1}}$

* Find the Laplace transform of $2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t$

June 2017

Soln: The given function be denoted by $f(t)$ and let,

$$f(t) = F(t) + G(t) + H(t)$$

where $F(t) = 2^t$,

$$G(t) = \frac{\cos 2t - \cos 3t}{t}$$

$$H(t) = t\sin t$$

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$\therefore \mathcal{L}[f(t)] = \mathcal{L}[F(t)] + \mathcal{L}[G(t)] + \mathcal{L}[H(t)]$ — ①

Now, $\mathcal{L}[F(t)] = \mathcal{L}[2^t]$

$$= \mathcal{L}[e^{\log 2 \cdot t}]$$

$$\mathcal{L}[F(t)] = \frac{1}{s - \log 2} \qquad \left[\because e^{\log 2} = 2\right]$$

So $e^{\log 2 \cdot t} = 2^t$

$$G(t) = \frac{\cos 2t - \cos 3t}{t}$$

We property ③ $\longrightarrow \mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\, ds$

$$\mathcal{L}[G(t)] = \int_s^{\infty} \bar{f}(s)\, ds$$

$$= \int_s^{\infty} \mathcal{L}[\cos 2t - \cos 3t]\, dt$$

$$= \int_s^{\infty} \left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] dt$$

$$= \left[\tfrac{1}{2}\log(s^2+4) - \tfrac{1}{2}\log(s^2+9)\right]_s^{\infty}$$

$$= \tfrac{1}{2}\left[\log(s^2+4) - \log(s^2+9)\right]_s^{\infty}$$

$$= \left\{\tfrac{1}{2} \log\left[\frac{s^2+4}{s^2+9}\right]\right\}_s^{\infty}$$

t sint

$$= \left[\log\left[\frac{s^2+4}{s^2+9}\right]^{1/2}\right]_s^{\infty}$$

$$= \left[\log\sqrt{\frac{s^2+4}{s^2+9}}\right]_s^{\infty}$$

$$\cancel{= \left[\log\sqrt{\frac{s^2(1+4/s^2)}{s^2(1+9/s^2)}}\right]_s^{\infty}}$$

$= \cancel{\log}$

$$= \lim_{s\to\infty} \log\sqrt{\frac{s^2+4}{s^2+9}} - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= \lim_{s\to\infty} \log\sqrt{\frac{\cancel{s^2}(1+4/s^2)}{\cancel{s^2}(1+9/s^2)}} - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= \log\sqrt{\frac{1+0}{1+0}} - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= \log(1) - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= 0 - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= \log\left(\sqrt{\frac{s^2+4}{s^2+9}}\right)^{-1}$$

$$= \log\left(\frac{1}{\sqrt{\frac{s^2+4}{s^2+9}}}\right)$$

$$\mathcal{L}[G(t)] = \log\left(\sqrt{\frac{s^2+9}{s^2+4}}\right)$$

$H(t) = t\sin t \longrightarrow \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$

use property ②

$$\mathcal{L}[H(t)] = \mathcal{L}[t\sin t] = (-1)\frac{d}{ds}(\bar{f}(s))$$

but

$$= -\frac{d}{ds}[\mathcal{L}(\sin t)]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{\cancel{(s^2+1)(0)} - (1)(2s)}{(s^2+1)^2}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

eq^n ① becomes

$$\mathcal{L}[f(t)] = \frac{1}{s-\log 2} + \log\sqrt{s^2+9}/s^2+4 + \frac{2s}{(s^2+1)^2}$$

* Find the Laplace transform of $t^2e^{-3t}\sin 2t$

Sol^n: we shall find first $\mathcal{L}(t^2\sin 2t)$

$$\mathcal{L}[t^2\sin 2t] = (-1)^2\frac{d^2}{ds^2}\mathcal{L}(\sin 2t)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left[\frac{2}{s^2+4}\right]\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+4)(0)-2(2s)}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$$

$$= \left[\frac{(s^2+4)^2(-4)-(-4s)(2)(s^2+4)(2s)}{((s^2+4)^2)^2}\right]$$

$$= \cancel{(s^2+4)}\left\{\frac{-4(s^2+4)+16s^2}{\cancel{(s^2+4)^4}}\right\}$$

$$= \frac{16s^2-4s^2-16}{(s^2+4)^3}$$

$$\mathcal{L}[t^2\sin 2t] = \frac{12s^2-16}{(s^2+4)^3}$$

$$\mathcal{L}[e^{-3t}t^2\sin 2t] = \left[\frac{12s^2-16}{(s^2+4)^3}\right]_{s\to s+3} \quad \{\text{property ①}\}$$

$$\mathcal{L}[e^{-3t}t^2\sin 2t] = \frac{12(s+3)^2-16}{((s+3)^2+4)^3}$$

Laplace transform of periodic functions

~~Statements~~

Definition: A function $f(t)$ is said to be a periodic function of period $T>0$ if $f(t+nT) = f(t)$ where $n = 1, 2, 3 \ldots$

ex: $\sin t$, $\cos t$ are periodic functions of period 2π because
$\sin(t+2n\pi) = \sin t$, $\cos(t+2n\pi) = \cos t$

Theorem: If $f(t)$ is a periodic function of period T, then $\boxed{L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt}$

[without proof]

Problems

① If $f(t) = t^2$, $0<t<2$ and $f(t+2) = f(t)$ for $t>2$, find $L[f(t)]$.

Soln: $f(t)$ is a periodic function of period 2.

$\therefore T = 2$

we have $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$L[f(t)] = \frac{1}{1-e^{-s2}} \int_0^2 e^{-st}\, t^2\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 t^2 \cdot e^{-st}\, dt$$

Apply Bernoulli's rule of integration by parts

$$\cancel{\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\left\{\left[t^2\frac{e^{-st}}{-s} - 2t\times\frac{1}{-s}\times\frac{e^{-st}}{-s} + 2\cdot\frac{1}{s^2}\times\frac{e^{-st}}{-s}\right]_0^2\right.}$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\left[t^2\frac{e^{-st}}{-s} - 2t\cdot\frac{1}{-s}\times\frac{e^{-st}}{-s} + 2\cdot\frac{1}{s^2}\times\frac{e^{-st}}{-s}\right]_0^2$$

$$= \frac{1}{1-e^{-2s}}\left[t^2\cdot\frac{e^{-st}}{-s} - \frac{2t\,e^{-st}}{s^2} - \frac{2}{s^3}e^{-st}\right]_0^2$$

$$= \frac{1}{1-e^{-2s}}\left[\left\{2^2\cdot\frac{e^{-2s}}{-s} + \frac{2(2)e^{-2s}}{s^2} - \frac{2}{s^3}e^{-2s}\right\} - \left\{0-0-\frac{2}{s^3}\right\}\right]$$

$\boxed{dt}$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{+s} + \frac{4e^{-2s}}{s^2} - \frac{2}{s^3}e^{-2s} + \frac{2}{s^3}\right] \rightarrow \text{Lcm is } s^3$$

$$= \frac{2}{s^3(1-e^{-2s})}\left[2e^{-2s}(-s^2) + 2e^{-2s}\cdot s - e^{-2s} + 1\right]$$

$$= \frac{2}{s^3(1-e^{-2s})}\left[e^{-2s}(-2s^2 + 2s - 1) + 1\right]$$

$$= \frac{2}{s^3(1-e^{-2s})}\left[1 - e^{-2s}(2s^2 + 2s + 1)\right]$$

(2) Find the Laplace transform of the full wave rectifier $f(t) = E\sin\omega t,\ 0 < t < \pi/\omega$ having period π/ω

Dec 2017, 18

Sol^n: we have $\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$f(t)$ is a periodic function of period π/ω

Now

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-s(\pi/\omega)}}\int_0^{\pi/\omega} e^{-st} E \sin\omega t \, dt.$$

$$\mathcal{L}[f(t)] = \frac{E}{1-e^{-\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$\because$ w.k.t $\displaystyle\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$\mathcal{L}[f(t)] = \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[\left\{e^{-s\pi/\omega}\left(-s\sin\omega\tfrac{\pi}{\omega} - \omega\cos\omega\tfrac{\pi}{\omega}\right) - e^{0}(0-\omega(1))\right\}\right]$$

$$= \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[\left\{e^{-s\pi/\omega}(-s\sin\pi - \omega\cos\pi) - (1)(-\omega)\right\}\right]$$

$$= \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[e^{-s\pi/\omega}(0-\omega(-1)) + \omega\right]$$

$$= \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[e^{-s\pi/\omega}(\omega) + \omega\right]$$

$$= \frac{E\omega}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[1+e^{-s\pi/\omega}\right]$$

$\times^{ly}$ both Nr & Dr by $e^{\pi s/2\omega}$ in RHS

$$= \frac{E\omega\,(1+e^{-s\pi/\omega})(e^{\pi s/2\omega})}{(1-e^{-\pi s/\omega})(s^2+\omega^2)\,e^{\pi s/2\omega}}$$

$$= \frac{E\omega}{s^2+\omega^2} \cdot \frac{e^{\pi s/2\omega} + e^{-s\pi/\omega} \cdot e^{\pi s/2\omega}}{e^{\pi s/2\omega} - e^{-\pi s/\omega} \cdot e^{\pi s/2\omega}}$$

$$= e^{-s\pi/\omega + \pi s/2\omega}$$
$$= e^{-\frac{2s\pi + \pi s}{2\omega}}$$
$$= e^{-s\pi/2\omega}$$

$$= \frac{E\omega}{s^2+\omega^2} \cdot \frac{e^{\pi s/2\omega} + e^{-s\pi/2\omega}}{e^{\pi s/2\omega} - e^{-s\pi/2\omega}}$$

$$\left[\text{W.K.T} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2} \right]$$

So X^{ly} and $\div$ both Nr & Dr by 2

$$\mathcal{L}[f(t)] = \frac{E\omega}{s^2+\omega^2} \times \frac{2\left(\frac{e^{\pi s/2\omega} + e^{-s\pi/2\omega}}{2}\right)}{2\left(\frac{e^{\pi s/2\omega} - e^{-s\pi/2\omega}}{2}\right)}$$

$$= \frac{E\omega}{s^2+\omega^2} \times \frac{\cancel{2}\cosh(\pi s/2\omega)}{\cancel{2}\sinh(\pi s/2\omega)}$$

$$\mathcal{L}[f(t)] = \frac{E\omega}{s^2+\omega^2} \coth(\pi s/2\omega)$$

(3) Given $f(t) = \begin{cases} E, & 0<t<a/2 \\ -E & a/2<t<a \end{cases}$ where $f(t+a) = f(t)$

Jan 2020 ACU

S.T $\mathcal{L}[f(t)] = E/s \cdot \tanh(as/4)$.

Solⁿ: The Given function is periodic with period $T = a$

we have $\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)\, dt$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-as}}\left\{\int_0^{a/2} e^{-st} E\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\int_0^{a/2} e^{-st}\,dt - \int_{a/2}^{a} e^{-st}\,dt\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} + \left[\frac{e^{-st}}{s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left[\left\{\frac{e^{-s\cdot a/2}}{-s} - \frac{e^{0}}{-s}\right\} + \left\{\frac{e^{-sa}}{s} - \frac{e^{-s\cdot a/2}}{s}\right\}\right]$$

$$= \frac{E}{1-e^{-as}}\left\{\frac{e^{-s\cdot a/2}}{-s} + \frac{1}{s} + \frac{e^{-sa}}{s} - \frac{e^{-sa/2}}{s}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left\{-e^{-sa/2} + 1 + e^{-sa} - e^{-sa/2}\right\}$$

$$= \frac{E}{s(1-e^{-sa})}\left\{-2e^{-sa/2} + 1 + e^{-sa}\right\}$$

$$= \frac{E}{s(1-e^{-sa})}\left\{1 + e^{-sa} - 2e^{-sa/2}\right\}$$

$\hookrightarrow a = 1,\ b = e^{-as/2}$ write it in the form of $(a-b)^2$

$$= \frac{E}{s(1-e^{-sa})}\left\{\left(1 - e^{-as/2}\right)^2\right\}$$

$$\mathcal{L}[f(t)] = \frac{E\,\cancel{(1-e^{-as/2})}^{2}}{s\,\cancel{(1-e^{-as/2})}(1+e^{-as/2})}$$

$$\mathcal{L}[f(t)] = \frac{E(1-e^{-as/2})}{s(1+e^{-as/2})}$$

$\times^{ly}$ both Nr & Dr by $e^{as/4}$ we get

$$\mathcal{L}[f(t)] = \frac{E(1-e^{-as/2})e^{as/4}}{s(1+e^{-as/2})e^{as/4}}$$

$$= \frac{E(e^{as/4} - e^{-as/2}\cdot e^{as/4})}{s(e^{as/4}+e^{-as/2}\,e^{as/4})}$$

$$= \frac{E(e^{as/4} - e^{-as/2+as/4})}{s(e^{as/4}+e^{-as/2+as/4})}$$

$$= e^{-as/2+as/4} = e^{\frac{-as\cdot 2+as}{4}} = e^{-as/4}$$

$$= \frac{E(e^{as/4}-e^{-as/4})}{s(e^{as/4}+e^{-as/4})}$$

$\times^{ly}$ & $\div$ by 2 both Nr & Dr

$$= \frac{E\,\cancel{2}\,\frac{(e^{as/4}-e^{-as/4})}{2}}{s\cdot\cancel{2}\,\frac{(e^{as/4}+e^{-as/4})}{2}}$$

w.k.T.

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$= \frac{E}{s}\times\frac{\sinh(as/4)}{\cosh(as/4)}$$

$$\mathcal{L}[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

Do yourself

(4) If $f(t) = \begin{cases} E, & 0<t<a \\ -E & a<t<2a \end{cases}$ S.T $\mathcal{L}[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{2}\right)$

H.W

June 2017

(5) If $f(t) = \begin{cases} t & 0 \le t \le a \\ 2a-t, & a \le t \le 2a, \end{cases}$ $f(t+2a) = f(t)$

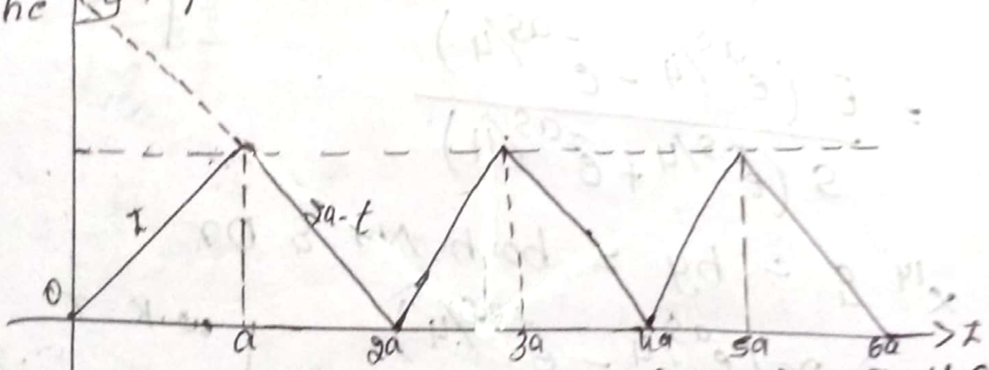
June & Dec 2016

(i) Sketch the graph of $f(t)$ as a periodic function

(ii) S.T $\mathcal{L}[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$

Sol^n: (i) Let $f(t) = y$ and $y = t$ is a straight line passing through the origin making an angle 45° with the t-axis. $y = 2a-t$ or $y + t = 2a$ or $t/2a + y/2a = 1$ is a straight line passing through the points $(2a, 0)$ and $(0, 2a)$.

The graph of $y = f(t)$ is as follows



The periodic function $f(t)$ is called triangular wave function.

(ii) we have $T = 2a$ and $\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a t e^{-st} dt + \int_a^{2a} (2a-t) e^{-st} dt \right\}$$

Apply Bernoulli's rule

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\left[t\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^a + \left[(2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\left[\left(-\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2}\right) - \left(0 - \frac{e^0}{s^2}\right)\right] + \left[\left((2a-2a)\frac{e^{-2as}}{s} - (-1)\frac{e^{-2as}}{s^2}\right) - \left((2a-a)\frac{e^{-as}}{-s} + \frac{e^{-sa}}{s^2}\right)\right]\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{-\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2} + 0 + \frac{e^{-2as}}{s^2} - \left(-\frac{ae^{-as}}{s} + \frac{e^{-sa}}{s^2}\right)\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\cancel{-\frac{ae^{-sa}}{s}} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \cancel{\frac{ae^{-as}}{s}} - \frac{e^{-sa}}{s^2}\right\}$$

$$= \frac{1}{(1-e^{-2as})s^2}\left\{-e^{-sa} + 1 + e^{-2as} - e^{-sa}\right\}$$

$$= \frac{1}{s^2(1-e^{-2as})}\left\{1 - 2e^{-sa} + e^{-2as}\right\}$$

$\hookrightarrow a = 1$

$b = e^{-as}$

write in the form of $(a-b)^2$

$$= \frac{1}{s^2(1-e^{-2as})}\left\{(1-e^{-as})^2\right\}$$

$$\mathcal{L}[f(t)] = \frac{\cancel{(1-e^{-as})}^2}{s^2\cancel{(1-e^{-as})}(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$\times^{ly}$ & $\div$ both Nr & Dr by $e^{as/2}$

$$= \frac{e^{as/2} - e^{-as}\, e^{as/2}}{s^2(e^{as/2} + e^{-as}\, e^{as/2})}$$

$$= \frac{e^{as/2} - e^{-as/2}}{s^2(e^{as/2} + e^{-as/2})}$$

$\times^{ly}$ & $\div$ both Nr & Dr by 2

$$= \frac{1}{s^2}\frac{\dfrac{\cancel{2}(e^{as/2} - e^{-as/2})}{2}}{\dfrac{\cancel{2}(e^{as/2} + e^{-as/2})}{2}}$$

$$= \frac{1}{s^2}\frac{\sinh(as/2)}{\cosh(as/2)}$$

$$= \frac{1}{s^2}\tanh(as/2)$$

$$\therefore \mathcal{L}[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

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Do yourself

June 2015

* Find $\mathcal{L}[f(t)]$ if $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \end{cases}$

June 2018

* Find $\mathcal{L}[f(t)]$ if $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$

⑥ A periodic function of period $2\pi/\omega$ is defined by $f(t) = \begin{cases} E \sin \omega t, & 0 \le t \le \pi/\omega \\ 0, & \pi/\omega \le t \le 2\pi/\omega \end{cases}$ where E and ω are constant. S.T $\mathcal{L}[f(t)] = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

Sol^n: we have for periodic function $f(t)$.

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad \text{Here } T = 2\pi/\omega$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\, dt$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \int_0^{\pi/\omega} e^{-st} E \sin \omega t\, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0 \cdot dt \right\}$$

$$\mathcal{L}[f(t)] = \frac{E}{1-e^{-2\pi s/\omega}} \left\{ \int_0^{\pi/\omega} e^{-st} \sin \omega t\, dt \right\}$$

w.k.T $\displaystyle\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

$$\mathcal{L}[f(t)] = \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}\left(-s\sin\omega t - \omega\cos\omega t\right)\right]_0^{\pi/\omega}$$

$$= \frac{E}{(s^2+\omega^2)\,1-e^{-2\pi s/\omega}}\left[\left\{e^{-s\pi/\omega}\left(-s\sin\omega\,\pi/\omega - \omega\cos\omega\,\pi/\omega\right)\right\} - \left\{e^{0}\,(0-\omega)\right\}\right]$$

$$= \frac{E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[-e^{-s\pi/\omega}\,(s\underbrace{\sin\pi}_{0} + \omega\cos\pi) + \omega\right]$$

$$= \frac{E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[-e^{-s\pi/\omega}\,(\omega(-1)) + \omega\right]$$

$$= \frac{E\omega}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[1 + e^{-s\pi/\omega}\right]$$

$$\begin{aligned}
&(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega}) \\
&= 1+\cancel{e^{-s\pi/\omega}} - \cancel{e^{-s\pi/\omega}} - e^{-s\pi/\omega}\cdot e^{-s\pi/\omega} \\
&= 1-e^{-2s\pi/\omega}
\end{aligned}$$

$$= \frac{E\omega\,\cancel{(1+e^{-s\pi/\omega})}}{(s^2+\omega^2)(1-e^{-s\pi/\omega})\cancel{(1+e^{-s\pi/\omega})}}$$

$$\mathcal{L}[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1+e^{-s\pi/\omega})}$$

Unit Step function (Heaviside function)

Definition :- The unit step function $u(t-a)$ or Heaviside function $H(t-a)$ is defined as follows

$$u(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$ where a is constant

Properties associated with the unit step function

① $\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$

② $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$ where $\mathcal{L}[f(t)] = \bar{f}(s)$

NOTE: ① when $f(t-a) = 1$ we have $f(t)$ also equal to 1 and hence $\mathcal{L}[f(t)] = 1/s$.

② $\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$

③ If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$

then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

④ If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$.

Problems:-

Working procedure

TYPE ①: To find $\mathcal{L}[F(t)u(t-a)]$ where $F(t)$ is a polynomial in t.

Step ①: Let $F(t) = f(t-a)$ which imply that $F(t+a) = f(t)$.

i.e Replace t by $t+a$ to obtain $f(t)$ and find $\mathcal{L}[f(t)] = \bar{f}(s)$

Step ②: $\mathcal{L}[F(t)\,u(t-a)] = \mathcal{L}[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$

by property (ii)

Type ②: Given $f(t)$ as discontineous function, to find $\mathcal{L}[f(t)]$ by expressing $f(t)$ in terms of unit step function

Step ①: we express $f(t)$ in terms of unit step function by directly making use of result (iii) or (iv) as the case may be

Step ②: we find $\mathcal{L}[f(t)]$ as in Type 1

Problems

* Find the Laplace transform of the following functions

$\{F(t)\,u(t-a)\}$ form

① $[e^{t-1} + \sin(t-1)]u(t-1)$

Sol^n: Let $F(t) = e^{t-1} + \sin(t-1)$

$F(t) = f(t-1)$

$f(t-1) = [e^{t-1} + \sin(t-1)]$

To get $f(t)$, replace t by $t+1$

$F(t+1) = f(t)$

$f(t) = e^{t} + \sin t$

$$\mathcal{L}[f(t)] = \mathcal{L}[e^{t} + \sin t]$$

$$= \mathcal{L}[e^{t}] + \mathcal{L}[\sin t]$$

$$= \frac{1}{s-1} + \frac{1}{s^2+1}$$

$$= \bar{f}(s)$$

$\mathcal{L}[f(t-1)u(t-1)] = e^{-s}\bar{f}(s)$ $(\because a=1)$

$\mathcal{L}[e^{t-1} + \sin(t-1)]u(t-1) = e^{-s}\left[\frac{1}{s-1} + \frac{1}{s^2+1}\right]$

② $\sin t\, u(t-\pi)$

$F(t) = \sin t$

Solⁿ: $F(t) = f(t-\pi)$

Do yourself

$f(t-\pi) = \sin t$

To get $f(t)$, replace t by $t+\pi$

$f(t) = \sin(t+\pi)$ ⟶ $\sin(180+\theta) = -\sin\theta$

$f(t) = -\sin t$ $\because$ 3rd quadrant $\sin\theta$ is $-ve$

$\mathcal{L}[f(t)] = \mathcal{L}[-\sin t]$

$= -1[\mathcal{L}\{\sin t\}]$

$\mathcal{L}[f(t)] = \frac{-1}{s^2+1}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\therefore \mathcal{L}[f(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{f}(s)$ $(\because a=\pi)$

$= e^{-\pi s} \times \frac{-1}{s^2+1}$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\bar{f}(s)$

$\mathcal{L}[f(t-\pi)u(t-\pi)] = -\frac{e^{-\pi s}}{s^2+1}$ $\Rightarrow \mathcal{L}[\sin t\, u(t-\pi)] = -\frac{e^{-\pi s}}{s^2+1}$

③ $(1-e^{+2t})u(t+1)$

$F(t) = 1-e^{2t}$

Solⁿ: $F(t) = f(t+1)$

$f(t+1) = 1-e^{+2t}$

To get $f(t)$ replace t by $t-1$

$f(t) = 1 - e^{+2(t-1)}$

$= 1 - e^{2t-2}$

$f(t) = 1 - e^{2t-2} \Rightarrow f(t) = 1 - e^{2t} \cdot e^{-2}$

$\mathcal{L}[f(t)] = \mathcal{L}[1 - e^{2t} \cdot e^{-2}]$

$= \mathcal{L}[1] - \mathcal{L}[e^{2t} \cdot e^{-2}]$

$= 1/s - e^{-2} \mathcal{L}[e^{2t}]$

$= 1/s - e^{-2} \dfrac{1}{s-2}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}[f(t+1)u(t+1)] = e^{s} \cdot \bar{f}(s) \quad (\because a = -1)$

$\mathcal{L}[(1-e^{2t})u(t+1)] = e^{s}\left(1/s - e^{-2}\left(\dfrac{1}{s-2}\right)\right)$

(4) $(3t^2 + 4t + 5)u(t-3)$

$f(t) = 3t^2 + 4t + 5$

Soln: $f(t) = f(t-3)$

$f(t-3) = 3t^2 + 4t + 5$

replace t by $t+3$

$f(t) = 3(t+3)^2 + 4(t+3) + 5$

$= 3[t^2 + 9 + 6t] + 4t + 12 + 5$

$= 3t^2 + 27 + 18t + 4t + 12 + 5$

$f(t) = 3t^2 + 22t + 44$

$\mathcal{L}[f(t)] = \mathcal{L}[3t^2 + 22t + 44]$

$= 3\mathcal{L}[t^2] + 22\mathcal{L}[t] + \mathcal{L}[44]$

$$= 3 \cdot \frac{2!}{s^3} + 22 \cdot \frac{1!}{s^2} + \frac{44}{s}$$

W.K.T

$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$$\mathcal{L}[f(t)] = \frac{6}{s^3} + \frac{22}{s^2} + \frac{44}{s}$$

$$\mathcal{L}[f(t)] = \bar{f}(s)$$

$$\mathcal{L}[f(t-3)\,u(t-3)] = e^{-3s}\,\bar{f}(s) \quad (\because a = 3)$$

$$\mathcal{L}[(3t^2+4t+5)\,u(t-3)] = e^{-3s}\left[\frac{6}{s^3} + \frac{22}{s^2} + \frac{44}{s}\right]$$

⑤ $(t^3+t^2+t+1)\,u(t+1)$

Soln: $F(t) = t^3+t^2+t+1$

$F(t) = f(t+1)$

$f(t+1) = t^3+t^2+t+1$

Replace t by $t-1$

$$f(t) = (t-1)^3 + (t-1)^2 + (t-1) + 1$$

Do yourself

$$= [\cancel{\text{Do yourself}}\; t^3 - 3t^2 + 3t] + [t^2 + \cancel{1} - 2t] + t - \cancel{1} + \cancel{1}$$

$$= t^3 + 2t - 2t^2 = t^3 - 2t^2 + 2t$$

$$\mathcal{L}[f(t)] = \mathcal{L}[t^3] - 2\mathcal{L}[t^2] + 2\mathcal{L}[t]$$

$$= \frac{3!}{s^4} - 2 \cdot \frac{2!}{s^3} + 2 \cdot \frac{1}{s^2}$$

$$\mathcal{L}[f(t)] = \frac{6}{s^4} - \frac{4}{s^3} + \frac{2}{s^2} = \bar{f}(s)$$

$$\mathcal{L}[(t^3+t^2+t+1)\,u(t+1)] = e^{s}\left[\frac{6}{s^4} - \frac{4}{s^3} + \frac{2}{s^2}\right]$$

$(\because a = -1)$

⑥ $(t^2-6t+9)\,e^{-(t-3)}\,u(t-3)$

Soln: $F(t) = f(t-3)$

$$f(t-3) = (t^2-6t+9)\,e^{-(t-3)}$$

$$= (t-3)^2\,e^{-(t-3)}$$

replace t by $t+3$

$$f(t) = (t)^2 e^{-t}$$

$$\mathcal{L}[f(t)] = \mathcal{L}[t^2 \cdot e^{-t}] \quad \{\text{property } ①\}$$

$$= \mathcal{L}[t^2]_{s \to s+1}$$

W.K.T $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$\mathcal{L}[t^2] = \frac{2!}{s^3}$

$$= \frac{2!}{(s+1)^3}$$

$$= \frac{2}{(s+1)^3}$$

$$\mathcal{L}[f(t)] = \bar{f}(s)$$

$$\mathcal{L}[f(t-3)u(t-3)] = e^{-3s}\bar{f}(s) \quad (\because a = 3)$$

$$= e^{-3s}\left\{\frac{2}{(s+1)^3}\right\}$$

$$\mathcal{L}\left[(t^2-6t+9)e^{-(t-3)} u(t-3)\right] = \frac{2e^{-3s}}{(s+1)^3}$$

* Express the following functions in terms of Heaviside unit step function and hence find their Laplace transform

① $f(t) = \begin{cases} t & 0 < t < 4 \\ 5 & t > 4 \end{cases}$

Sol^n: $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$
(by property iii)

$$f(t) = t + [5-t]u(t-4)$$

$$\mathcal{L}[f(t)] = \mathcal{L}(t) + \mathcal{L}[(5-t)u(t-4)] \quad — ①$$

we have $\mathcal{L}(t) = \frac{1}{s^2}$

$f(t-4) = 5-t$

Replace t by $t+4$ to get $f(t)$

$f(t) = 5-(t+4)$

$f(t) = 5-t-4$

$f(t) = 1-t$

$\mathcal{L}[f(t)] = \mathcal{L}[1] - \mathcal{L}[t]$

$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{1}{s^2}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}[f(t-4)\, u(t-4)] = e^{-4s}\bar{f}(s) \quad (\because a=4)$

$\mathcal{L}[(5-t)\, u(t-4)] = e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$

use these results in ①

$\mathcal{L}[f(t)] = \frac{1}{s^2} + e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$

$= \frac{1}{s}e^{-4s} + \frac{1}{s^2} - e^{-4s}\frac{1}{s^2}$

$\mathcal{L}[f(t)] = \frac{1}{s}e^{-4s} + \frac{1}{s^2}(1-e^{-4s})$

② $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \sin t, & t>\pi \end{cases}$

Dec 2018

Soln: $f(t) = \cos t + (\sin t - \cos t)\, u(t-\pi)$ (by a property ③)

$\mathcal{L}[f(t)] = \mathcal{L}(\cos t) + \mathcal{L}[[\sin t - \cos t]\, u(t-\pi)]$ — ①

Now let $f(t-\pi) = \sin t - \cos t$

To get $f(t)$ replace ...

$f(t) = \sin(t+\pi) - \cos(t+\pi)$ $\qquad \sin(t+\pi) = -\sin t$

$= -\sin t - (-\cos t)$ $\qquad t+\pi$ lies in III qd

$f(t) = -\sin t + \cos t$ $\qquad \cos(t+\pi) = -\cos t$

$\mathcal{L}[f(t)] = \mathcal{L}[-\sin t + \cos t]$ $\qquad t+\pi$ lies in III qd

$= \mathcal{L}[-\sin t] + \mathcal{L}[\cos t]$ $\qquad$ In 3rd qd sin and cos functions are -ve

$$= -\frac{1}{s^2+1} + \frac{s}{s^2+1}$$

$$= \frac{-1+s}{s^2+1}$$

$$\mathcal{L}[f(t)] = \frac{s-1}{s^2+1}$$

$$\mathcal{L}[f(t-\pi)u(t-\pi)] = e^{-\pi s}F(s) \qquad (\because a = \pi)$$

$$\mathcal{L}[(\sin t - \cos t)u(t-\pi)] = \frac{e^{-\pi s}(s-1)}{s^2+1}$$

put this in ①

$$\mathcal{L}[f(t)] = \mathcal{L}[\cos t] + \frac{e^{-\pi s}(s-1)}{s^2+1}$$

$$= \frac{s}{s^2+1} + \frac{e^{-\pi s}(s-1)}{s^2+1}$$

$$\mathcal{L}[f(t)] = \frac{s + e^{-\pi s}(s-1)}{s^2+1} \;//$$

③ June 2017

$$f(t) = \begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$$

Sol^n:

$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

$f(t) = \sin t + [\cos t - \sin t]u(t - \pi/2)$ (by property ②)

$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] + \mathcal{L}[(\cos t - \sin t)u(t - \pi/2)]$ — ①

$\mathcal{L}[\sin t] = 1/(s^2+1)$

$f(t-\pi/2) = \cos t - \sin t$

To get $f(t)$ replace t by $t+\pi/2$

$f(t) = \cos(t+\pi/2) - \sin(t+\pi/2)$

$f(t) = -\sin t - \cos t$

$\mathcal{L}[f(t)] = \mathcal{L}[-\sin t - \cos t]$

$= \mathcal{L}[-\sin t] + \mathcal{L}[-\cos t]$

$= \frac{-1}{s^2+1} - \frac{s}{s^2+1}$

$= \frac{-1-s}{s^2+1}$

$= \frac{-(s+1)}{s^2+1}$

$\cos(\pi/2 + t)$
$\hookrightarrow (90+\theta)$
$= -\sin t$
$\because$ 90+t in II q
cos function is -ve

$\sin(\pi/2 + t)$
$= \cos t$
90+t lie in II q
sin function is +ve

$\mathcal{L}[f(t-\pi/2)\, u(t-\pi/2)] = e^{-\pi s/2} F(s)$

$\mathcal{L}[(\cos t - \sin t) u(t-\pi/2)] = e^{-\pi s/2} \frac{(-(s+1))}{s^2+1}$

use this result in ①

$\mathcal{L}[f(t)] = \frac{1}{s^2+1} - \frac{e^{-\pi s/2}(s+1)}{s^2+1}$

$\mathcal{L}[f(t)] = \frac{1 - e^{-\pi s/2}(s+1)}{s^2+1}$

④ Do yourself $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

Ans: $\mathcal{L}[f(t)] = \frac{2}{s^2+4} - \frac{2e^{-\pi s}}{s^2+4}$

$= \frac{2(1-e^{-\pi s})}{s^2+4}$

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(6) June 2016, 18

$$f(t) = \begin{cases} 1 & 0 < t \le 1 \\ t & 1 < t \le 2 \\ t^2 & t > 2 \end{cases}$$

soln: by property (14)

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$$

$$f(t) = 1 + [t-1]u(t-1) + [t^2 - t]u(t-2) \quad (8)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[1] + \mathcal{L}[(t-1)u(t-1)] + \mathcal{L}[(t^2-t)u(t-2)] \quad \text{—— (1)}$$

W.K.T $\mathcal{L}[1] = 1/s$

$\mathcal{L}[(t-1)u(t-1)]$

$\Rightarrow F(t-1) = t-1$

To get $F(t)$ replace t by $t+1$

$F(t) = t$

$\mathcal{L}[F(t)] = \mathcal{L}[t] = 1!/s^2$ | $\mathcal{L}[t^n] = n!/s^{n+1}$

$\mathcal{L}[F(t)] = 1/s^2$

$\mathcal{L}[F(t)] = F(s)$

$\mathcal{L}[F(t-1)u(t-1)] = e^{-s}F(s)$

$\Rightarrow \mathcal{L}[(t-1)u(t-1)] = 1/s^2 \times e^{-s}$

$\mathcal{L}[(t^2-t)u(t-2)]$

$\Rightarrow G(t-2) = t^2 - t$

To get $G(t)$ replace t by $t+2$

$G(t) = (t+2)^2 - (t+2)$

$= t^2 + 4 + 4t - t - 2$

$G(t) = t^2 + 3t + 2$

$\mathcal{L}[G(t)] = \mathcal{L}[t^2] + 3\mathcal{L}[t] + \mathcal{L}[2]$

$\mathcal{L}[2]$ or $2\mathcal{L}[1] = 2/s$

$= \frac{2!}{s^3} + 3 \cdot \frac{1!}{s^2} + 2/s$

$= 2/s^3 + 3/s^2 + 2/s$

$\mathcal{L}[G(t)] = \bar{G}(s) \Rightarrow \mathcal{L}[G(t-2)\,u(t-2)] = e^{-2s}\bar{G}(s)$

$\mathcal{L}[(t^2 - t)\,u(t-2)] = e^{-2s}\,\bar{G}(s)$

$\mathcal{L}[(t^2 - t)\,u(t-2)] = e^{-2s}\left[2/s^3 + 3/s^2 + 2/s\right]$

put these results in ①

$\mathcal{L}[f(t)] = 1/s + e^{-s}/s^2 + e^{-2s}\left(2/s^3 + 3/s^2 + 2/s\right)$

Do yourself

⑤ June 2018

$$f(t) = \begin{cases} \cos t, & 0 < t \le \pi \\ 1 & \pi < t \le 2\pi \\ \sin t & t > 2\pi \end{cases}$$

Sol^n: by property ④

$f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a) + [f_3(t) - f_2(t)]\,u(t-b)$

$f(t) = \cos t + [1 - \cos t]\,u(t-\pi) + [\sin t - 1]\,u(t-2\pi)$

$\mathcal{L}[f(t)] = \mathcal{L}[\cos t] + \mathcal{L}[(1 - \cos t)\,u(t-\pi)] + \mathcal{L}[(\sin t - 1)\,u(t-2\pi)]$ —— ①

③ ⑤ Continued $\mathcal{L}[\cos t] = \frac{s}{s^2+1}$

$\mathcal{L}[(1-\cos t)u(t-\pi)]$

$\Rightarrow$ Let $F(t-\pi) = 1-\cos t$

To get $F(t)$ replace t by $t+\pi$

$F(t) = 1-\cos(t+\pi)$

$= 1-(-\cos t))$

$F(t) = 1+\cos t$

$\cos(t+\pi) = -\cos t$

$\downarrow$

lying in III Qr

cos function is −ve in III Qr

$\mathcal{L}[F(t)] = \mathcal{L}[1+\cos t]$

$= \mathcal{L}[1] + \mathcal{L}[\cos t]$

$\mathcal{L}[F(t)] = \frac{1}{s} + \frac{s}{s^2+1}$

$\mathcal{L}[F(t)] = \bar{F}(s)$

$\mathcal{L}[F(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{F}(s) \quad (\because a = \pi)$

$\mathcal{L}[(1-\cos t)u(t-\pi)] = e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right)$

$\mathcal{L}[(\sin t - 1)u(t-2\pi)]$

$\Rightarrow$ Let $G(t-2\pi) = \sin t - 1$

To get $G(t)$ replace t by $t+2\pi$

$G(t) = \sin(t+2\pi) - 1$

$G(t) = \sin t - 1$

$\sin(2\pi+t) = \sin t$

$2\pi+t$ lies in I Qr

In I Qr all trigonometric functions are +ve

$\mathcal{L}[G(t)] = \mathcal{L}[\sin t - 1]$

$= \mathcal{L}[\sin t] - \mathcal{L}[1]$

$\mathcal{L}[G(t)] = \frac{1}{s^2+1} - \frac{1}{s}$

$\mathcal{L}[G(t)] = \bar{G}(s)$

$$L[G_1(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\,\bar{G}_1(s)$$

$$L[(\sin t - 1)u(t-2\pi)] = e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s}\right]$$

put all these results in ①

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)$$

Do yourself

* $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$ (use property (iv))

Ans: $L[f(t)] = \frac{s}{s^2+1} + s e^{-\pi s}\left(\frac{1}{s^2+4} + \frac{1}{s^2+1}\right) - \frac{5 s e^{-2\pi s}}{(s^2+4)(s^2+9)}$

* $f(t) = \begin{cases} e^{2t} & 0 < t < 1 \\ 2, & t > 1 \end{cases}$

(use property (iii))

Ans: $L[f(t)] = \frac{1}{s-2} + e^{-s}\left(\frac{2}{s} - \frac{e^2}{s-2}\right)$

⑧ Express the function $f(t) = \begin{cases} \pi - t, & 0 < t \le \pi \\ \sin t, & t > \pi \end{cases}$ in terms of unit step function and hence find its laplace transform

Dec 2017

Soln: by the property of unit step function

$$f(t) = f_1(t) + [f_2(t) - f_1(t)\,u(t-a)]$$

$$f(t) = \pi - t + [\sin t - (\pi - t)]u(t-\pi)]$$

$$L[f(t)] = L[\pi - t] + L[(\sin t - \pi + t)\,u(t-\pi)] \quad \text{—①}$$

$$L[\pi - t] = L[\pi] - L[t]$$

$$= \pi L[1] - L[t]$$

$$= \pi \cdot \frac{1}{s} - \frac{1!}{s^2} \qquad \left(\because L[t^n] = \frac{n!}{s^{n+1}}\right)$$

$$L[\pi - t] = \frac{\pi}{s} - \frac{1}{s^2}$$

$f(t-\pi) = \sin t - \pi + t$

to get $f(t)$ replace t by $t+\pi$

$f(t) = \sin(t+\pi) - \cancel{\pi} + t + \cancel{\pi}$

$f(t) = -\sin t + t$

$\mathcal{L}[f(t)] = \mathcal{L}[-\sin t + t]$

$= \mathcal{L}[-\sin t] + \mathcal{L}[t]$

$= \frac{-1}{s^2+1} + \frac{1!}{s^2}$

$\mathcal{L}[f(t)] = \frac{-1}{s^2+1} + \frac{1}{s^2}$

$\mathcal{L}[f(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{f}(s)$

$\mathcal{L}[(\sin t - \pi + t)u(t-\pi)] = e^{-\pi s}\left(\frac{-1}{s^2+1} + \frac{1}{s^2}\right)$

put they all in ①

$\mathcal{L}[f(t)] = \left(\frac{\pi}{s} - \frac{1}{s^2}\right) + e^{-\pi s}\left(\frac{-1}{s^2+1} + \frac{1}{s^2}\right)$

$= \left(\frac{\pi}{s} - \frac{1}{s^2}\right) + e^{-\pi s}\left(\frac{-\cancel{s^2} + \cancel{s^2} + 1}{s^2(s^2+1)}\right)$

$\mathcal{L}[f(t)] = \left(\frac{\pi}{s} - \frac{1}{s^2}\right) + e^{-\pi s}\left(\frac{1}{s^2(s^2+1)}\right)$

(2) Define heaviside unit step function. using unit step function find Laplace transform of $f(t) = \begin{cases} \sin t & 0 \le t < \pi \\ \sin 2t & \pi \le t < 2\pi \\ \sin 3t & t \ge 2\pi \end{cases}$

Dec 2016

Sol^n:

write the definition of unit step function

by using the standard property

$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]$
$u(t-b)$

$f(t) = \sin t + [(\sin 2t - \sin t)]u(t-\pi) +$
$[\sin 3t - \sin 2t]u(t-2\pi)$

$L[f(t)] = L[\sin t] + L[(\sin 2t - \sin t)u(t-\pi)]$
$+ L[(\sin 3t - \sin 2t)u(t-2\pi)]$ —— (1)

$L[\sin t] = 1/(s^2+1)$

Let $f(t-\pi) = \sin 2t - \sin t$

To get $F(t)$ replace t by $t+\pi$

$F(t) = \sin 2(t+\pi) - \sin(t+\pi)$
$= \sin(2t+2\pi) - \sin(t+\pi)$

$360+2t$ i.e $(360+\theta)$ $\to$ I Q

$180+t$ $\to$ III Q

$= \sin 2t - (-\sin t)$

$F(t) = \sin 2t + \sin t$

$L[F(t)] = L[\sin 2t] + L[\sin t]$

$= 2/(s^2+4) + 1/(s^2+1)$

$\mathcal{L}[F(t)] = \bar{F}(s)$

$\mathcal{L}[F(t-\pi)\,u(t-\pi)] = e^{-\pi s}\,\bar{F}(s)$

$\mathcal{L}\{[\sin 2t - \sin t]\,u(t-\pi)\} = e^{-\pi s}\left\{\frac{2}{s^2+4} + \frac{1}{s^2+1}\right\}$

Let $G(t-2\pi) = \sin 3t - \sin 2t$

To get $G(t)$ replace t by $t+2\pi$

$G(t) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$

$= \sin(3t+6\pi) - \sin(2t+2\pi)$

$\sin(360+2t)$

$= \sin 2t$

$\sin(3(2\pi)+3t)$

$= \sin 3t$

$G(t) = \sin 3t - \sin 2t$

$\mathcal{L}[G(t)] = \mathcal{L}[\sin 3t - \sin 2t]$

$= \mathcal{L}[\sin 3t] - \mathcal{L}[\sin 2t]$

$\mathcal{L}[G(t)] = \frac{3}{s^2+9} - \frac{2}{s^2+4}$

$\mathcal{L}[G(t)] = \bar{G}(s)$

$\mathcal{L}[G(t-2\pi)\,u(t-2\pi)] = e^{-2\pi s}\,\bar{G}(s)$

$\mathcal{L}[(\sin 3t - \sin 2t)\,u(t-2\pi)] = e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$

Inverse Laplace Transforms

If $\mathcal{L}[f(t)] = \bar{f}(s)$, then $f(t)$ is called inverse Laplace transform of $\bar{f}(s)$ & is denoted by $\mathcal{L}^{-1}[\bar{f}(s)]$

Thus we can say that

$$\mathcal{L}[f(t)] = \bar{f}(s) \Leftrightarrow \mathcal{L}^{-1}[\bar{f}(s)] = f(t)$$

eg: $\mathcal{L}(1) = \frac{1}{s} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

NOTE:

① $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$

② $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

③ $\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$.

④ $\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

⑤ $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

⑥ $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

⑦ $\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$

⑧ $\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

⑨ $\mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$

⑩ $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] (n>-1) = \frac{t^n}{\Gamma(n+1)}$

⑪ $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right]$ $n = 1, 2, 3 \ldots = \frac{t^n}{n!}$

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Note: ① $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$L^{-1}\left[\frac{1}{s-1}\right] = e^{t}$

② $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$L^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$

③ $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$L^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t$

④ $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

$L^{-1}\left[\frac{3}{s^2+9}\right] = \sin 3t$

⑤ $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$L^{-1}\left[\frac{s}{s^2-16}\right] = \cosh 4t$

⑥ $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

$L^{-1}\left[\frac{1}{s^2+5}\right] = \frac{1}{\sqrt{5}} \sin(\sqrt{5}\, t)$

$(\sqrt{5})^2$

⑦ $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$

$L^{-1}\left[\frac{4}{s^2-16}\right] = \sinh 4t$

⑧ $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$

$L^{-1}\left[\frac{1}{s^2-36}\right] = \frac{\sinh 6t}{6}$

⑨ $\mathcal{L}^{-1}\left(\frac{1}{s^{3/2}}\right) = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} = 2\sqrt{t/\pi}$

⑩ $\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{3!}$

$\mathcal{L}^{-1}[c_1\bar{f}(s) + c_2\bar{g}(s)] = c_1\mathcal{L}^{-1}[\bar{f}(s)] + c_2\mathcal{L}^{-1}[\bar{g}(s)]$

Find the inverse Laplace transform of the following

① $\frac{1}{s+2} + \frac{3}{2s+5} - \frac{4}{3s-2}$

Soln: $= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{3}{2s+5}\right] - \mathcal{L}^{-1}\left[\frac{4}{3s-2}\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{2s+5}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{3s-2}\right]$

$= e^{-2t} + 3\mathcal{L}^{-1}\left[\frac{1}{2(s+(5/2))}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{3(s-2/3)}\right]$

$= e^{-2t} + \frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{s+5/2}\right] - \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2/3}\right]$

$= e^{-2t} + \frac{3}{2}e^{-5/2\,t} - \frac{4}{3}e^{2/3\,t}$

② $\frac{2s-5}{4s^2+25} + \frac{8-6s}{16s^2+9}$

Soln: $\frac{2s}{4s^2+25} - \frac{5}{4s^2+25} + \frac{8}{16s^2+9} - \frac{6s}{16s^2+9}$

$2\mathcal{L}^{-1}\left[\frac{s}{4s^2+25}\right] - 5\mathcal{L}^{-1}\left[\frac{1}{4s^2+25}\right] + 8\mathcal{L}^{-1}\left[\frac{1}{16s^2+9}\right] - 6\mathcal{L}^{-1}\left[\frac{s}{16s^2+9}\right]$

$2\mathcal{L}^{-1}\left[\frac{s}{4(s^2+\frac{25}{4})}\right] - 5\mathcal{L}^{-1}\left[\frac{1}{4(s^2+\frac{25}{4})}\right] + 8\mathcal{L}^{-1}\left[\frac{1}{16(s^2+9/16)}\right] - 6\mathcal{L}^{-1}\left[\frac{s}{16(s^2+\frac{9}{16})}\right]$

$\frac{2}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2+(\frac{5}{2})^2}\right] - \frac{5}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2+(\frac{5}{2})^2}\right] + \frac{8}{16}\mathcal{L}^{-1}\left[\frac{1}{s^2+(3/4)^2}\right] - \frac{6}{16}\mathcal{L}^{-1}\left[\frac{s}{(s^2+(\frac{3}{4})^2)}\right]$

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$$\frac{1}{2}\cos\left(\tfrac{5}{2}\right)t - \frac{5}{4}\cdot\frac{\sin(5/2)t}{5/2} + \frac{1}{2}\,\frac{\sin(3/4)t}{3/4}$$

$$-\frac{3}{8}\times\cos\left(\tfrac{3}{4}\right)t$$

$$\frac{1}{2}\cos\left(\tfrac{5}{2}\right)t - \frac{2}{4\cdot 2}\sin\left(\tfrac{5}{2}\right)t + \frac{1}{2}\times\frac{4}{3}\sin\left(\tfrac{3}{4}\right)t - \frac{3}{8}\cos\left(\tfrac{3}{4}\right)t$$

$$\frac{1}{2}\cos\left(\tfrac{5}{2}\right)t - \frac{1}{2}\sin\left(\tfrac{5}{2}\right)t + \frac{2}{3}\sin\left(\tfrac{3}{4}\right)t - \frac{3}{8}\cos\left(\tfrac{3}{4}\right)t$$

$$\frac{1}{2}\cos\left(\tfrac{5}{2}\right)t - \frac{1}{2}\sin\left(\tfrac{5}{2}\right)t$$

Do yourself

(3) $\dfrac{s+2}{s^2+36} + \dfrac{4s-1}{s^2+25}$ Ans: $\cos 6t + \frac{1}{3}\sin 6t + 4\cos 5t - \frac{1}{5}\sin 5t$

(4) $\dfrac{2s-5}{8s^2-50} + \dfrac{4s}{9-s^2}$ Ans: $\frac{1}{4}e^{-5t/2} - 4\cosh 3t$

Ans: $= \dfrac{2s}{8s^2-50} - \dfrac{5}{8s^2-50} + \dfrac{4s}{9-s^2}$

$$= \frac{2s}{2(4s^2-25)} - \frac{5}{2(4s^2-25)} + \frac{4s}{9-s^2}$$

$$= \frac{s}{4(s^2-25/4)} - \frac{5}{2\times 4(s^2-\frac{25}{4})} - \frac{4s}{s^2-9}$$

$$= \frac{s}{4(s^2-(5/2)^2)} - \frac{5}{8(s^2-(5/2)^2)} - \frac{4s}{s^2-9}$$

$$= \frac{1}{4}L^{-1}\left[\frac{s}{s^2-(5/2)^2}\right] - \frac{5}{8}L^{-1}\left[\frac{1}{s^2-(5/2)^2}\right] - 4L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$= \frac{1}{4}\cosh\frac{5t}{2} - \frac{5}{8}\,\frac{\sinh\frac{5t}{2}}{5/2} - 4\cosh 3t$$

$$= \frac{1}{4}\cosh\tfrac{5}{2}t - \frac{1}{4}\sinh\tfrac{5}{2}t - 4\cosh 3t$$

$$= \frac{1}{4}\left\{\frac{e^{5/2t}+e^{-5/2t}}{2} - \frac{e^{5/2t}+e^{-5/2t}}{2}\right\} - 4\cosh 3t$$

$$= \frac{1}{4}\left\{\frac{2e^{-5/2t}}{2}\right\} - 4\cosh 3t$$

$$= \frac{1}{4}e^{-5/2t} - 4\cosh 3t$$

(4) $\dfrac{2s-5}{8s^2-50} + \dfrac{4s}{9-s^2}$

$$= L^{-1}\left[\frac{2s-5}{2(4s^2-25)}\right] + 4L^{-1}\left[\frac{s}{9-s^2}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{2s-5}{(2s)^2-5^2}\right] - 4L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{\cancel{2s-5}}{(2s+5)\cancel{(2s-5)}}\right] - 4\cosh 3t$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{2s+5}\right] - 4\cosh 3t$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{2(s+5/2)}\right] - 4\cosh 3t$$

$$= \frac{1}{2}\cdot\frac{1}{2}\,e^{-\frac{5t}{2}} - 4\cosh 3t$$

$$= \frac{1}{4}e^{-5t/2} - 4\cosh 3t$$

(5) $\dfrac{(s+2)^3}{s^6}$

Soln: we $(a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$$(s+2)^3 = s^3+8+6s^2+12s$$

$$\frac{(s+2)^3}{s^6} = \frac{s^3+8+6s^2+12s}{s^6}$$

$$= \frac{1}{s^3} + \frac{8}{s^6} + \frac{6}{s^4} + \frac{12}{s^5}$$

$$= L^{-1}\left[\frac{1}{s^3}\right] + L^{-1}\left[\frac{8}{s^6}\right] + L^{-1}\left[\frac{6}{s^4}\right] + L^{-1}\left[\frac{12}{s^5}\right]$$

$$= L^{-1}\left[\frac{1}{s^3}\right] + 8L^{-1}\left[\frac{1}{s^6}\right] + 6L^{-1}\left[\frac{1}{s^4}\right] + 12L^{-1}\left[\frac{1}{s^5}\right]$$

$$= \frac{t^2}{2!} + 8\cdot\frac{t^5}{5!} + 6\cdot\frac{t^3}{3!} + 12\cdot\frac{t^4}{4!}$$

$$= \frac{t^2}{2} + 8\cdot\frac{t^5}{120} + 6\frac{t^3}{6} + 12\cdot\frac{t^4}{24}$$

$$L^{-1}\left[\frac{(s+2)^3}{s^6}\right] = \frac{t^2}{2} + \frac{t^5}{15} + t^3 + \frac{t^4}{2}$$

* Do yourself

* $\frac{3(s^2-1)^2}{2s^5}$

Ans: $\frac{3}{2}\left[1 - t^2 + \frac{t^4}{24}\right]$

Computation of the inverse transform of $e^{-as}\bar{f}(s)$

W.K.T $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$$\mathcal{L}^{-1}[e^{-as}\bar{f}(s)] = f(t-a)u(t-a)$$

Working procedure:

① In the given function we should observe the presence of e^{-as} first and identify the remaining part of the function to be called as $\bar{f}(s)$.

② Taking the inverse of $\bar{f}(s)$ we obtain $f(t)$

③ The required inverse of $e^{-as}\bar{f}(s)$ is obtained by replacing t by $t-a$ in $f(t)$ to be multiplied by unit step function $u(t-a)$.

⊛ Find the inverse Laplace transform of the following

① $\dfrac{1+e^{-3s}}{s^2}$

Soln: $\mathcal{L}^{-1}\left[\dfrac{1+e^{-3s}}{s^2}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2}\right]$$

$$= t + (t-3)u(t-3)$$

$\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s^2}\right] = (t-3)u(t-3)$

$\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right] = t$

$$\therefore \mathcal{L}^{-1}\left[\frac{1+e^{-3s}}{s^2}\right] = t + (t-3)u(t-3)$$

② $\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}$

Sol^n: $\mathcal{L}^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right]$

$= \mathcal{L}^{-1}\left[\frac{3}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{2e^{-s}}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{3e^{-2s}}{s}\right]$

$= 3\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 2\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^3}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s}\right]$ —— ①

$\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = \frac{t}{1!} = t$

$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{1+1}}\right] = \frac{t^1}{1!} = t$ ∵ $n=1$

$\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!} = \frac{t^2}{2}$

$\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{2+1}}\right] = \frac{t^2}{2}$

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{0+1}}\right] = \frac{t^0}{0!} =$

hence ① becomes

$\mathcal{L}^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right]$

$= 3t + 2\cdot\frac{(t-1)^2}{2}u(t-1) - 3(1)u(t-2)$

$\mathcal{L}^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right] = 3t + (t-1)^2u(t-1) - 3u(t-2)$

③ Find the inverse Laplace transform of

$\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+4}$

Sol^n: $\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{se^{-2\pi s}}{s^2+4}\right]$ —①

$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$

$\mathcal{L}[\cos 2t] = \frac{s}{s^2+4}$

W.K.T. $\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$

$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$

(1) becomes

$= \sin(t-\pi)u(t-\pi) + \cos 2(t-2\pi)u(t-2\pi)$

$= -\sin t\, u(t-\pi) + \cos 2t\, u(t-2\pi)$

$L^{-1}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+1}\right] = -\sin t\, u(t-\pi) + \cos 2t\, u(t-2\pi)$

$\sin(t-\pi)$
$= \sin(-(\pi - t))$
w.k.t $\sin(-\theta) = -\sin\theta$
$= -\sin(\pi - t)$ (2^{nd} Q)
$= -\sin t$

* Do yourself

(4) $\dfrac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}$

Ans: $L^{-1}\left[\frac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right] = \sin\pi t\, u(t-\tfrac{1}{2}) - \sin\pi t\, u(t-1)$

common

$L^{-1}\left[\frac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right] = \sin\pi t\left[u(t-\tfrac{1}{2}) - u(t-1)\right]$

(5) $\dfrac{\cosh 2s}{e^{3s}\cdot s^2}$

Soln: $\dfrac{\cosh 2s}{e^{3s}\cdot s^2} = \dfrac{\cosh 2s}{s^2} \times e^{-3s}$

$= \dfrac{e^{-3s}}{s^2}\left[\dfrac{e^{2s} + e^{-2s}}{2}\right]$

$= \dfrac{1}{2}\left[\dfrac{e^{-s} + e^{-5s}}{s^2}\right]$

$\dfrac{\cosh 2s}{e^{3s}\cdot s^2} = \dfrac{1}{2}\left[\dfrac{e^{-s}}{s^2} + \dfrac{e^{-5s}}{s^2}\right]$

$L^{-1}\left[\dfrac{\cosh 2s}{e^{3s}\cdot s^2}\right] = \dfrac{1}{2}\left\{L^{-1}\left[\dfrac{e^{-s}}{s^2}\right] + L^{-1}\left[\dfrac{e^{-5s}}{s^2}\right]\right\}$

W.K.T $L^{-1}\left[\frac{1}{s^2}\right] = t$

$L^{-1}\left[\frac{\cosh 2s}{e^{3s} \cdot s^2}\right] = \frac{1}{2}\left\{(t-1)u(t-1) + (t-5)u(t-5)\right\}$

⑥ Do yourself

$$\frac{(1-e^{-s})(2-e^{-2s})}{s^3}$$

Soln: $\frac{(1-e^{-s})(2-e^{-2s})}{s^3} = \frac{2 - e^{-2s} - 2e^{-s} + e^{-s} \cdot e^{-2s}}{s^3}$

$$= \frac{2 - e^{-2s} - 2e^{-s} + e^{-3s}}{s^3}$$

$$L^{-1}\left[\frac{(1-e^{-s})(2-e^{-2s})}{s^3}\right] = L^{-1}\left[\frac{2 - e^{-2s} - 2e^{-s} + e^{-3s}}{s^3}\right]$$

$$= L^{-1}\left[\frac{2}{s^3}\right] - L^{-1}\left[\frac{e^{-2s}}{s^3}\right] - L^{-1}\left[\frac{2e^{-s}}{s^3}\right] + L^{-1}\left[\frac{e^{-3s}}{s^3}\right]$$

$$= 2L^{-1}\left[\frac{1}{s^3}\right] - L^{-1}\left[\frac{e^{-2s}}{s^3}\right] - 2L^{-1}\left[\frac{e^{-s}}{s^3}\right] + L^{-1}\left[\frac{e^{-3s}}{s^3}\right]$$

$$= 2 \cdot \frac{t^2}{2!} - \frac{(t-2)^2}{2!}u(t-2) - 2\frac{(t-1)^2}{2!}u(t-1) + \frac{(t-3)^2}{2!}u(t-3)$$

$$= t^2 - \frac{(t-2)^2 u(t-2)}{2} - (t-1)^2 u(t-1) + \frac{(t-3)^2 u(t-3)}{2}$$

$$= t^2 - (t-1)^2 u(t-1) - \frac{(t-2)^2 u(t-2)}{2} + \frac{(t-3)^2 u(t-3)}{2}$$

** Inverse transform by completing the square

we have property that if $\mathcal{L}[f(t)] = \bar{f}(s)$

then $\mathcal{L}[e^{at} f(t)] = \bar{f}(s-a) \Rightarrow \mathcal{L}^{-1}[\bar{f}(s)] = f(t)$ - ①

$\Rightarrow \mathcal{L}^{-1}[\bar{f}(s-a)] = e^{at} f(t)$ - ②

from ① and ② we have

$\mathcal{L}^{-1}[\bar{f}(s-a)] = e^{at} \mathcal{L}^{-1}[\bar{f}(s)]$ - ③

working procedure:

① Given $\bar{f}(s) = \dfrac{\phi(s)}{ps^2+qs+r}$

we first express (ps^2+qs+r) in the form $(s-a)^2 \pm b^2$ and later express $\phi(s)$ in terms of $(s-a)$ so that given function of s reduces to a function $s-a$

② we use ③ to obtain the result

③ However if $\bar{f}(s) = \phi(s)/\varphi(s-a)$ we only need to express $\phi(s)$ in terms of $(s-a)$ to find inverse transform.

* Find the inverse Laplace transform of the following functions

① $\dfrac{s+5}{s^2-6s+13}$

Soln: $\mathcal{L}^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right] = \mathcal{L}^{-1}\left[\dfrac{s+5}{s^2-2\times 3s+13}\right]$

$a=s \quad b=3$

$$= \mathcal{L}^{-1}\left[\frac{s+5+3-3}{s^2-6s+9-9+13}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s+\bar{3}+5+3}{(s-3)^2-9+13}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-3+8}{(s-3)^2+4}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{(s-3)+8}{(s-3)^2+2^2}\right]$$

here $a=3$ and $s-3$ changy to s

$$= e^{3t}\,\mathcal{L}^{-1}\left[\frac{s+8}{s^2+2^2}\right]$$

$$= e^{3t}\left\{\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right]+\mathcal{L}^{-1}\left[\frac{8}{s^2+2^2}\right]\right\}$$

$$= e^{3t}\left\{\cos 2t + \cancel{8}4\,\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right]\right\}$$

$$= e^{3t}\left\{\cos 2t + 4.\sin 2t\right\}$$

$$\mathcal{L}^{-1}\left[\frac{s+5}{s^2-6s+13}\right] = e^{3t}\left\{\cos 2t + 4\sin 2t\right\}$$

② $\cancel{\frac{(s+2)e^{-s}}{(s+1)^4}}$

Soln: $\cancel{\mathcal{L}^{-1}\left[\frac{(s+2)e^{-s}}{(s+1)^4}\right]} =$

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② $\frac{s+1}{s^2+6s+9}$

Sol^n: $\mathcal{L}^{-1}\left[\frac{s+1}{s^2+6s+9}\right] = \mathcal{L}^{-1}\left[\frac{(s+3)-2}{(s+3)^2}\right]$

here $a=-3$ & $s-3$ change to s

$$= e^{-3t}\mathcal{L}^{-1}\left[\frac{s-2}{s^2}\right]$$

$$= e^{-3t}\left\{\mathcal{L}^{-1}\left[\frac{s}{s^2}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\right\}$$

$$= e^{-3t}\left\{\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\right\}$$

$$= e^{-3t}\{1-2.t\}$$

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+6s+9}\right] = e^{-3t}(1-2t)$$

③ $\frac{e^{-4s}}{(s-4)^2}$

Sol^n: Let $\bar{f}(s) = \frac{1}{(s-4)^2}$

$\mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{1}{(s-4)^2}\right]$ $\because a=4$ and $s-4$ changes to s

$$= e^{4t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{4t}.t$$

$\mathcal{L}^{-1}[\bar{f}(s)] = f(t)$

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{(s-4)^2}\right] = f(t-4)u(t-4)$$

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{(s-4)^2}\right] = \left\{e^{4(t-4)}(t-4)\right\}u(t-4)$$

⑧ $\dfrac{(s+2)e^{-s}}{(s+1)^4}$

Sol^n: $\bar{f}(s) = \dfrac{s+2}{(s+1)^4}$

we shall first find $L^{-1}[\bar{f}(s)] = f(t)$

$$L^{-1}\left[\frac{s+2}{(s+1)^4}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^4}\right]$$

here $a=-1$, $s+1$ changes to s

$$= e^{-t} L^{-1}\left[\frac{s+1}{s^4}\right]$$

$$= e^{-t}\left\{L^{-1}\left(\frac{s}{s^4}\right) + L^{-1}\left(\frac{1}{s^4}\right)\right\}$$

$$= e^{-t}\left\{L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^4}\right)\right\}$$

$$= e^{-t}\left\{\frac{t^2}{2!} + \frac{t^3}{3!}\right\}$$

$$= e^{-t}\left\{\frac{t^2}{2} + \frac{t^3}{6}\right\}$$

$$L^{-1}[e^{-s}\bar{f}(s)] = f(t-1)u(t-1)$$

$$L^{-1}\left[e^{-s}\frac{s+2}{(s+1)^4}\right] = e^{-(t-1)}\left\{\frac{(t-1)^2}{2} + \frac{(t-1)^3}{6}\right\}u(t-1)$$

Inverse transform by the method of partial fractions

W.K.T the method of partial fraction is a technique of converting an algebraic function $\phi(s)$ in to a sum. $\psi(s)$

Depending on the nature of terms in $\psi(s)$ we have to split into a sum of various terms with constants A, B, C, D... which can be determined. Later the inverse is found term by term.

* $\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$

* $\dfrac{1}{s(s+1)^2} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$

* $\dfrac{1}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}$

* Find the inverse Laplace transform of the following functions

① $\dfrac{1}{s(s+1)(s+2)(s+3)}$

Sol^n:

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3} \quad \text{— ①}$$

$\times^{ly}$ B.S by $s(s+1)(s+2)(s+3)$

$$1 = A(s+1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s+1)(s+3) + D(s)(s+1)(s+2)$$

put $s=0$

$1 = A(1)(2)(3) + 0 + 0 + 0$

$1 = 6A$

$A = 1/6$

put $s = -1$

$1 = A(0)(1)(2) + B(-1)(1)(2) + C(-1)(0)(2) + D(-1)(0)(1)$

$1 = 0 - 2B + 0 + 0$

$1 = -2B$

$B = -1/2$

put $s = -2$

$1 = A(0) + B(0) + C(-2)(-1)(1) + D(0)$

$1 = 0 + 0 + 2C \pm 0$

$1 = 2C$

$C = 1/2$

put $s = -3$

$1 = A(0) + B(0) + C(0) + D(-3)(-2)(-1)$

$1 = 0 + 0 + 0 - 6D$

$1 = -6D$

$D = -1/6$

① $\Rightarrow$

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{1/6}{s} + \frac{-1/2}{s+1} + \frac{1/2}{s+2} + \frac{-1/6}{s+3}$$

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{1}{6s} - \frac{1}{2(s+1)} + \frac{1}{2(s+2)} - \frac{1}{6(s+3)}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

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$$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t}$$

② Find $L^{-1}\left[\frac{3s+2}{s^2-s-2}\right]$

s^2-s-2

$s^2-2s+s-2$

$s(s-2)+1(s-2)$

$(s-2)(s+1)$

Sol: $\frac{3s+2}{s^2-s-2} = \frac{3s+2}{(s-2)(s+1)}$

$$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \quad -①$$

$\times^{14}$ B.S by $(s-2)(s+1)$

$3s+2 = A(s+1) + B(s-2)$

put $s=2$

$3(2)+2 = A(2+1) + B(0)$

$8 = 3A$

$A = 8/3$

put $s=-1$

$3(-1)+2 = A(0) + B(-3)$

$+1 = +3B$

$B = 1/3$

put A and B in ①

$$\frac{3s+2}{(s-2)(s+1)} = \frac{8/3}{s-2} + \frac{1/3}{s+1}$$

$$\frac{3s+2}{(s-2)(s+1)} = \frac{8}{3(s-2)} + \frac{1}{3(s+1)}$$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3} L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3} L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3} e^{2t} + \frac{1}{3} e^{-t}$$

(3) $\dfrac{s+2}{s^2(s+3)}$

Sol: $\dfrac{s+2}{s^2(s+3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3}$

x^{ly} B.S by $s^2(s+3)$

$s+2 = As(s+3) + B(s+3) + Cs^2$ — (1)

$s+2 = A(s^2+3s) + B(s+3) + Cs^2$

co-eff of s^2 is A

put $s=0$

$2 = 0 + B(3) + 0$

$3B = 2$

$B = 2/3$

put $s = -3$

$-3+2 = As(0) + B(0) + 9C$

$-1 = 0 + 0 + 9C$

$9C = -1$

$C = -1/9$

equating the co-efficient of s^2 on both sidy of (1)

$0 = A + C$

$A = -C$

$A = -(-1/9)$

$A = 1/9$

put A, B, C values in (1)

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1/9}{s} + \frac{2/3}{s^2} + \frac{-1/9}{s+3}$$

$$\frac{s+2}{s^2(s+3)} = \frac{1}{9s} + \frac{2}{3s^2} + \frac{1}{9(s+3)}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9}L^{-1}\left[\frac{1}{s}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{9}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{9}(1) + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9}(1 + e^{-3t}) + \frac{2}{3}t$$

(4) $\dfrac{4s+5}{(s+1)^2(s+2)}$

Solⁿ: $\dfrac{4s+5}{(s+1)^2(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2}$ — (1)

$\times^{ly}$ B.S by $(s+1)^2(s+2)$

$4s+5 = A(s+1)(s+2) + B(s+2) + C(s+1)^2$ — (*)

put $s=-1$

$4(-1)+5 = A(0) + B(-1+2) + C(0)$

$1 = 0 + B + 0$

$\therefore B = 1$

put $s=-2$

$4(-2)+5 = A(-2+1)(0) + B(0) + C(-2+1)^2$

$-3 = 0 + 0 + C(-1)^2$

$-3 = C$

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$\therefore \underline{C = -3}$

Equating the Co-efficient of S^2 on both sidy of ⊛ we get.

$0 = A + C$

$0 = A - 3$

$\underline{A = 3}$

put A. B. C valuy in ①

$$\frac{4S+5}{(S+1)^2(S+2)} = \frac{3}{S+1} + \frac{1}{(S+1)^2} + \frac{-3}{S+2}$$

$$\mathcal{L}^{-1}\left[\frac{4S+5}{(S+1)^2(S+2)}\right] = 3\mathcal{L}^{-1}\left[\frac{1}{S+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(S+1)^2}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{S+2}\right]$$

$$= 3e^{-t} + e^{-t}\mathcal{L}^{-1}[1/S^2] - 3e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{4S+5}{(S+1)^2(S+2)}\right] = 3e^{-t} + e^{-t}.t - 3e^{-2t}$$

⑧ $\dfrac{S+2}{S^2(S+3)}$

Soln: $\dfrac{S+2}{S^2(S+3)} = \dfrac{A}{S} + \dfrac{B}{S^2} + \dfrac{C}{S+3}$ — ①

$\times^{ly}$ B.S by $S^2(S+3)$

$S+2 = AS(S+3) + B(S+3) + CS^2$ — ⊛

put $S=0$

$2 = 0 + 3B + 0 \Rightarrow 3B = 2 \Rightarrow B = 2/3$

put $S = -3$

$-3+2 = A(0) + 0 + C(-3)^2$

$-1 = 9C$

$\underline{C = -1/9}$

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equating the Co-efficient of s^2 on both sidey of ⊛ we get

$$0 = A + C$$

$$\therefore A = -C = -(-1/9)$$

$$A = 1/9$$

put A, B, C values in ①

$$\frac{s+2}{s^2(s+3)} = \frac{1/9}{s} + \frac{2/3}{s^2} + \frac{-1/9}{s+3}$$

$$\frac{s+2}{s^2(s+3)} = \frac{1}{9s} + \frac{2}{3}\cdot\frac{1}{s^2} + \frac{-1}{9(s+3)}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = 1/9\, L^{-1}\left[1/s\right] + 2/3\, L^{-1}\left[1/s^2\right] - 1/9\, L^{-1}\left[\frac{1}{s+3}\right]$$

$$= 1/9(1) + 2/3 \cdot t - 1/9\, e^{-3t}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = 1/9 + 2/3\, t - 1/9\, e^{-3t} //$$

⑥ $\frac{(3s+1)\,e^{-3s}}{(s-1)(s^2+1)}$

Soln: $\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$ — ①

$\times^{ly}$ B.S by $(s-1)(s^2+1)$

$$3s+1 = A(s^2+1) + (Bs+C)(s-1) \quad — ⊛$$

put $s = 1$; $4 = A(1+1) + (B(1)+C)(0)$

$$4 = 2A + 0$$

$$2A = 4$$

$$A = 2$$

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put $s=0$

$1 = A(1) + (B(0)+C)(0-1)$

$1 = A + (0+C)(-1)$

$1 = A + (C)(-1)$

$1 = A - C$

W.K.T $A = 2$

$1 = 2 - C$

$1 - 2 = -C$

$-1 = -C$

$C = 1$

equating the co-effficient of s^2 on B.S of (1) we get

$0 = A + B$

$\therefore B = -A$

$B = -2$

(1) $\Rightarrow$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$

$$\frac{3s+1}{(s-1)(s^2+1)} = \left[\frac{2}{s-1}\right] + \left[\frac{-2s}{s^2+1}\right] + \left[\frac{1}{s^2+1}\right]$$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2e^t - 2\cos t + \sin t$$

$$L^{-1}\left[\frac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}\right] = \left[2e^{t-3} - 2\cos(t-3) + \sin(t-3)\right]u(t-3)$$

* Inverse transform of logarithmic functions and inverse functions

Given $\bar{f}(s)$ we need to find $L^{-1}[\bar{f}(s)] = f(t)$

we have the property $L[f(t)] = -\bar{f}'(s)$ — ①

equivalently, $L^{-1}[-\bar{f}'(s)] = t f(t)$.

* Find the inverse laplau transform of the following functions :-

① $\log\left[\frac{s+a}{s+b}\right]$

Sol^{n}: let $\bar{f}(s) = \log\left[\frac{s+a}{s+b}\right]$

$\log(m/n) = \log m - \log n$

$\log(mn) = \log m + \log n$

$\bar{f}(s) = \log(s+a) - \log(s+b)$

D.w.r. to s

$\bar{f}'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

x^{ly} B.S by -1

$-\bar{f}'(s) = -\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$

$-\bar{f}'(s) = \frac{1}{s+b} - \frac{1}{s+a}$

$L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{1}{s+b} - \frac{1}{s+a}\right]$

$t f(t) = L^{-1}\left[\frac{1}{s+b}\right] - L^{-1}\left[\frac{1}{s+a}\right]$

$t f(t) = e^{-bt} - e^{-at}$

$f(t) = \frac{e^{-bt} - e^{-at}}{t}$

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② $\cot^{-1}(s/a)$

Solⁿ: Let $\bar{f}(s) = \cot^{-1}(s/a)$

D.w.r.to s

$$\bar{f}'(s) = \frac{-1}{1+(s/a)^2} \times \frac{d}{ds}(s/a)$$

$$= \frac{-1}{1+s^2/a^2} \times \frac{1}{a}$$

$$= \frac{-1}{\frac{a^2+s^2}{a^2}} \times \frac{1}{a}$$

$$= \frac{-\cancel{a^2}}{a^2+s^2} \times \frac{1}{\cancel{a}}$$

$$\bar{f}'(s) = \frac{-a}{a^2+s^2}$$

$\times^{ly}$ B.s by -1

$$-\bar{f}'(s) = \frac{a}{a^2+s^2} \Rightarrow -\bar{f}'(s) = \frac{a}{s^2+a^2}$$

$$L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$t\,f(t) = \sin at$$

$$f(t) = \frac{\sin at}{t}$$

③ $\log\left[\frac{s^2+4}{s(s+4)(s-4)}\right]$

Solⁿ: let $\bar{f}(s) = \log\left[\frac{s^2+4}{s(s+4)(s-4)}\right]$

$\log m \quad - \log n$

$$\bar{f}(s) = [\log(s^2+4)] - [\log\{s(s+4)(s-4)\}]$$

$\log mnp = \log m + \log n + \log p$

$$= \log(s^2+4) - \{\log s + \log(s+4) + \log(s-4)\}$$

$$\bar{f}(s) = \log(s^2+4) - \log s - \log(s+4) - \log(s-4)$$

D. w. r. to s

$$\bar{f}'(s) = \frac{1}{s^2+4}\frac{d}{ds}(s^2+4) - \frac{1}{s} - \frac{1}{s+4} - \frac{1}{s-4}$$

$$\bar{f}'(s) = \frac{1}{s^2+4} \times 2s - \frac{1}{s} - \frac{1}{s+4} - \frac{1}{s-4}$$

$\times^{ly}$ B.S by -1

$$-\bar{f}'(s) = -\frac{2s}{s^2+4} + \frac{1}{s} + \frac{1}{s+4} + \frac{1}{s-4}$$

$$L^{-1}[-\bar{f}'(s)] = -2L^{-1}\left[\frac{s}{s^2+4}\right] + L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{1}{s-4}\right]$$

$L^{-1}\left[\frac{s}{s^2+2^2}\right]$

$$t f(t) = -2\cos 2t + 1 + e^{-4t} - e^{4t}$$

$$f(t) = \frac{1 + e^{-4t} - e^{4t} - 2\cos 2t}{t}$$

(4) $\cot^{-1}\left(\frac{s+a}{b}\right)$

Sol^n: Let $\bar{f}(s) = \cot^{-1}\left(\frac{s+a}{b}\right)$

$$\bar{f}'(s) = \frac{-1}{1+\left(\frac{s+a}{b}\right)^2} \times \frac{d}{ds}\left(\frac{s+a}{b}\right)$$

$$\bar{f}'(s) = \frac{-1}{1+\frac{(s+a)^2}{b^2}} \times \left[\frac{b(1+0) - (s+a)(0)}{b^2}\right]$$

$$= \frac{-b^2}{b^2+(s+a)^2}\left[\frac{b-0}{\cancel{b^2}}\right]$$

$$\bar{f}'(s) = \frac{-b}{(s+a)^2+b^2}$$

$\times^{ly}$ B.S by $-$

$$-\bar{f}'(s) = \frac{b}{(s+a)^2+b^2}$$

$$\mathcal{L}^{-1}\left[-\bar{f}'(s)\right] = \mathcal{L}^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$$

$$t f(t) = e^{-at}\mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$t f(t) = e^{-at}\sin bt$$

$$f(t) = \frac{e^{-at}\sin bt}{t}$$

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Convolution theorem

Definition: The convolution of two functions $f(t)$ & $g(t)$ usually denoted by $f(t) * g(t)$ is defined in the form of an integral as follows

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

Property ①: $f(t) * g(t) = g(t) * f(t)$ is exist.

i.e to say that convolution operation * is commutative.

Convolution Theorem:

Statement: If $\mathcal{L}^{-1}[\bar{f}(s)] = f(t)$ and

$\mathcal{L}^{-1}[\bar{g}(s)] = g(t)$

then $$\mathcal{L}^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

Computation of the inverse transform by using convolution theorem:-

Working procedure:

① The given function is expressed as product of two functions say $\bar{f}(s)$ & $\bar{g}(s)$

② we find $\mathcal{L}^{-1}[\bar{f}(s)] = f(t)$ & $\mathcal{L}^{-1}[\bar{g}(s)] = g(t)$

③ we apply convolution theorem in one of the form.

$$\mathcal{L}^{-1}[\bar{f}(s)\, \bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

④ we evaluate the convolution integral to obtain the required inverse.

Using Convolution theorem obtain the inverse Laplace transform of the following functions.

① $\frac{1}{s(s^2+a^2)}$

Jan 2016
2018

Soln: Let $\bar{f}(s) = 1/s$ & $\bar{g}(s) = 1/(s^2+a^2)$

Taking inverse

$L^{-1}[\bar{f}(s)] = L^{-1}[1/s] = 1$ and $L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right]$

$\therefore f(t) = 1$ $\qquad g(t) = \frac{1}{a}\sin at$

we have convolution theorem

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_{u=0}^{t} 1\cdot\frac{\sin a(t-u)}{a}\,du$$

$$= \int_{u=0}^{t} \frac{\sin(at-au)}{a}\,du$$

$$= \frac{1}{a}\left[\int_{u=0}^{t} \sin(at-au)\,du\right]$$

$$= \frac{1}{a}\left[\frac{-\cos(at-au)}{-a}\right]_{u=0}^{t}$$

$$= \left[\frac{\cos(at-au)}{a^2}\right]_{u=0}^{t}$$

$$= \frac{1}{a^2}(\cos(at-a\cancel{t}) - \cos at)$$

$$= \frac{1}{a^2}(\cos(0) - \cos at)$$

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$= \frac{1}{a^2}(1-\cos at)$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}(1-\cos at)$$

(5) $\frac{s}{(s^2+a^2)^2}$

Dec 2016

Soln: Let $\bar{f}(s) = \frac{1}{s^2+a^2}$ $\bar{g}(s) = \frac{s}{s^2+a^2}$

$$\mathcal{L}^{-1}[\bar{f}(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$f(t) = \frac{\sin at}{a}$$

and $\mathcal{L}^{-1}[\bar{g}(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right]$

$$g(t) = \cos at$$

we have convolution theorem

$$\mathcal{L}^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\times\frac{s}{s^2+a^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a}\cdot\cos(at-au)\,du$$

w.k.T $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

where $A = au$, $B = at-au$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{a}\int_{u=0}^{t} \frac{1}{2}\left[\sin(au+at-au) + \sin(au-(at-au))\right]du$$

$$= \frac{1}{2a}\int_{u=0}^{t}(\sin at + \sin(2au-at))\,du$$

w.k.T ~~$\sin(-\theta) = -\sin\theta$~~

$$\cancel{= \frac{1}{2}\int_{u=0}^{t}(\sin at - \sin at)\,du}$$

$$= \frac{1}{2a}\int_{u=0}^{t}(\sin at + \sin(2au-at))\,du$$

$$= \frac{1}{2a}\left[\int_{u=0}^{t}\sin at\,du + \int_{u=0}^{t}\sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left\{\left[+\sin at\int_{u=0}^{t}1\,du\right] + \left[-\frac{\cos(2au-at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2a}\left[\sin at\,[u]_{u=0}^{t} - \left\{\frac{\cos(2at-at)}{2a} - \frac{\cos(0-at)}{2a}\right\}\right]$$

$$= \frac{1}{2a}\left[\sin at(t-0) - \frac{\cos(at)}{2a} + \frac{\cos(-at)}{2a}\right]$$

$$= \frac{1}{2a}\left[\sin at\,(t) - \cancel{\frac{\cos at}{2a}} + \cancel{\frac{\cos at}{2a}}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$$

③ June 2017 $\dfrac{1}{(s^2+a^2)^2}$

Soln: Let $\bar{f}(s) = \dfrac{1}{s^2+a^2}$ & $\bar{g}(s) = \dfrac{1}{s^2+a^2}$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] \ \& \ L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$f(t) = \frac{\sin at}{a}, \quad g(t) = \frac{\sin at}{a}$$

we have convolution theorem

$$L^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)} \times \frac{1}{(s^2+a^2)}\right] = \int_{u=0}^{t} \frac{\sin au}{a} \times \frac{\sin a(t-u)}{a}\,du$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a} \times \frac{\sin(at-au)}{a}\,du$$

$$= \frac{1}{a^2}\int_{u=0}^{t} \sin au \,\sin(at-au)\,du$$

W.K.T $\quad -2\sin A \sin B = \cos(A+B) - \cos(A-B)$

$$\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$= \frac{1}{a^2}\int_{u=0}^{t} -\frac{1}{2}\left[\cos(au+at-au) - \cos(au-(at-au))\right] du$$

(H)

Dec 2017

Sol

$$= -\frac{1}{2a^2}\int_{u=0}^{t} (\cos at - \cos(2au - at))\, du$$

$$= -\frac{1}{2a^2}\left[\int_{u=0}^{t} \cos at \cdot du - \int_{u=0}^{t} \cos(2au - at)\, du\right]$$

$$= -\frac{1}{2a^2}\left[\cos at \int_{u=0}^{t} 1 \cdot du - \left[\frac{\sin(2au - at)}{2a}\right]_{u=0}^{t}\right]$$

$$= -\frac{1}{2a^2}\left[\cos at \, [u]_{u=0}^{t} - \left\{\frac{\sin(2at - at)}{2a} - \frac{\sin(0 - at)}{2a}\right\}\right]$$

$$= -\frac{1}{2a^2}\left[\cos at\, [t - 0] - \left\{\frac{\sin at}{2a} - \frac{\sin(-at)}{2a}\right\}\right]$$

$$= -\frac{1}{2a^2}\left[t\cos at - \left\{\frac{\sin at}{2a} - \times - \frac{\sin at}{2a}\right\}\right]$$

$$= -\frac{1}{2a^2}\left[t\cos at - \frac{\sin at}{2a} - \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2a^2}\left[-t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2a^2}\left[-t\cos at + \frac{\cancel{2}\sin at}{\cancel{2}a}\right]$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[-t\cos at + \sin at\right]$$

(H) $\frac{1}{(s-1)(s^2+1)}$

Dec 2017

Soln: let $\bar{f}(s) = \frac{1}{s-1}$ & $\bar{g}(s) = \frac{1}{s^2+1}$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s-1}\right] = e^t$$

$$\therefore f(t) = e^t$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

$$\therefore g(t) = \sin t$$

Now we convolution theorem

$$L^{-1}[\bar{f}(s)\bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_{u=0}^{t} e^{u} \sin(t-u)\, du$$

w.k.T $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$

$a=1 \quad b=-1$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \left[\frac{e^u}{1+1}\left((1)\sin(t-u) - (-1)\cos(t-u)\right)\right]_{u=0}^{t}$$

$$= \left[\frac{e^u}{2}\left(\sin(t-u) + \cos(t-u)\right)\right]_{u=0}^{t}$$

$$= \left[\left\{\frac{e^t}{2}\left(\sin(t-t) + \cos(t-t)\right)\right\} - \left\{\frac{e^0}{2}\left(\sin(t-0) + \cos(t-0)\right)\right\}\right]$$

$$= \left\{\frac{e^t}{2}\left(\sin(0) + \cos(0)\right) - \frac{1}{2}(\sin t + \cos t)\right\}$$

$$= \frac{e^t}{2}(0+1) - \frac{1}{2}(\sin t + \cos t)$$

$$= \frac{1}{2}e^t - \frac{1}{2}\sin t - \frac{1}{2}\cos t$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{1}{2}\left(e^t - \sin t - \cos t\right)$$

Doyourself

(5) $\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$ using convolution theorem

June 2018

(6) $\frac{s^2}{(s^2+a^2)^2}$

Soln: Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{s}{s^2+a^2}$

$\mathcal{L}^{-1}[\bar{f}(s)] = \cos at$ and $\mathcal{L}^{-1}[\bar{g}(s)] = \cos at$

$f(t) = \cos at$ and $g(t) = \cos at$

Now by applying convolution theorem we have

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

$$= \int_{u=0}^{t} \cos au \cos a(t-u)\, du$$

$$= \int_{u=0}^{t} \cos au \cos(at-au)\, du$$

NOTE: $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$= \frac{1}{2}\int_{u=0}^{t} \left\{\cos(au+at-au) + \cos\left(au-(at-au)\right)\right\} du$$

$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+ \dots\right.$

$$= \frac{1}{2}\int_{u=0}^{t} \left(\cos at + \cos(au - at + au)\right) du$$

$$= \frac{1}{2}\int_{u=0}^{t} \left(\cos at + \cos(2au - at)\right) du$$

$$= \frac{1}{2}\left[\int_{u=0}^{t} \cos at \, du + \int_{u=0}^{t} \cos(2au - at)\, du\right]$$

$$= \frac{1}{2}\left\{\cos at \int_{u=0}^{t} 1 \, du + \int_{u=0}^{t} \cos(2au - at)\, du\right\}$$

$$= \frac{1}{2}\left\{\cos at \,[u]_{u=0}^{t} + \cancel{\times}\left[+\frac{\sin(2au - at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2}\left\{\cos at\,(t - 0) + \frac{1}{2a}\left\{\sin(2at - at) - \sin(2a(0) - at)\right\}\right\}$$

$$= \frac{1}{2}\left\{\cos at \cdot t + \frac{1}{2a}\left(\sin(at) - \sin(0 - at)\right)\right\}$$

$$= \frac{1}{2}\left\{t \cos at + \frac{1}{2a}\left(\sin at - \sin(-at)\right)\right\}$$

$$= \frac{1}{2}\left\{t \cos at + \frac{1}{2a}\left(\sin at + \sin at\right)\right\} \qquad \left[\begin{array}{l} \text{w.k.t} \\ \sin(-\theta) \\ = -\sin\theta \end{array}\right.$$

$$= \frac{1}{2}\left\{t \cos at + \frac{1}{\cancel{2}a}\left(\cancel{2}\sin at\right)\right\}$$

$$= \frac{1}{2}\left\{t \cos at + \frac{1}{a}\sin at\right\}$$

$\boxed{at - au))\}du}$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\left\{t\cos at + \frac{1}{a}\sin at\right\} = \frac{1}{2a}\left\{at \cos at + \sin at\right\}$$

Do yourself

(7) $\dfrac{1}{s^2(s+1)^2}$

Solⁿ: Let $\bar{f}(s) = 1/s^2$ $\quad \bar{g}(s) = 1/(s+1)^2$

$$L^{-1}[\bar{f}(s)] = L^{-1}[1/s^2] \text{ and } L^{-1}[\bar{g}(s)] = 1/(s+1)^2$$

$$f(t) = t \quad \text{and } g(t) = e^{-t}\cdot t$$

Now by applying Convolution theorem we have

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

$$= \int_{u=0}^{t} u\, e^{-(t-u)} (t-u)\, du$$

$$= \int_{u=0}^{t} u\cdot e^{-t+u}\cdot (t-u)\, du$$

$$= \int_{u=0}^{t} u\cdot e^{-t}\cdot e^{u} (t-u)\, du$$

$$= e^{-t}\int_{u=0}^{t} (tu-u^2)\, e^{u}\, du$$

Apply Bernoulli's rule

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = e^{-t}\left[(tu-u^2)e^{u} - (t-2u)e^{u} + (0-2)e^{u}\right]_{u=0}^{t}$$

$$= e^{-t}\left[\{(\cancel{t\cdot t}^{\,0} - t^2)e^{t} - (t-2t)e^{t} + (-2)e^{t}\} - \{(0-0)e^{0} - (t-0)e^{0} + (-2)e^{0}\}\right]$$

$= e^{-t}\left[0 + (-t)e^{t} - 2e^{t} - 0 + t + 2\right]$

$= e^{-t}\left[\cancel{t}e^{t} - 2e^{t} + t + 2\right]$

$= e^{-t}\cancel{t}.e^{t} - 2e^{t}.e^{-t} + te^{-t} + 2e^{-t}$

$= \cancel{t} - 2 + e^{-t}(2+t)$

(or)

$L^{-1}\left[\frac{1}{s^2(s+1)}\right] = 2(e^{-t} - 1) + t(1 + e^{-t})$

Solution of linear differential equations using Laplace transforms (Initial value problems)

Laplace transform of the derivatives

we derive an expression for $L[y'(t)]$ and hence we deduce the expression for $L[y''(t)]$, $L[y'''(t)]$

so, $L[y'(t)] = sL[y(t)] - y(0)$

$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$

$L[y'''(t)] = s^3L[y(t)] - s^2y(0) - sy'(0) - y''(0)$

working procedure

1. The given differential eqn is expressed in the notation $y'(t), y''(t), y'''(t)\ldots$ for the derivatives
2. we take laplace transform on both sides of given equation
3. we use the expressions for $L[y'(t)], L[y''(t)]\ldots$

④ we substitute the given initial conditions and simplify to obtain $\mathcal{L}[y(t)]$ as a function of s.

⑤ we find the inverse to obtain $y(t)$

Problems

① Solve by using Laplace transforms

$$\frac{d^2y}{dt^2} + k^2 y = 0 \quad \text{given that } y(0)=2,\ y'(0)=0$$

Dec 2018

Soln: The given eqn is $y''(t) + k^2 y(t) = 0$

Taking Laplace transform on B.S

$$\mathcal{L}[y''(t) + k^2 y(t)] = \mathcal{L}[0]$$

~~Taking Laplace~~

$$\{s^2 \mathcal{L}[y(t)] - s y(0) - y'(0)\} + k^2 \mathcal{L}[y(t)] = 0$$

using the given initial conditions we obtain

$$(s^2+k^2)\mathcal{L}[y(t)] - s y(0) - y'(0) = 0$$

$$(s^2+k^2)\mathcal{L}[y(t)] - s(2) - 0 = 0$$

$$(s^2+k^2)\mathcal{L}[y(t)] - 2s = 0$$

$$\mathcal{L}[y(t)] = \frac{2s}{s^2+k^2}$$

$$\mathcal{L}[y(t)] = 2\mathcal{L}^{-1}\left[\frac{s}{s^2+k^2}\right]$$

W.K.T $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$$\therefore y(t) = 2\cos kt$$

② Solve $y''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$ by using Laplace transform method.

Sol^{n}: $y''' + 2y'' - y' - 2y = 0$ — ①

Given initial condition $y(0) = y'(0) = 0$ & $y''(0) = 6$

Taking Laplace transform on B.S of ①

$$\mathcal{L}[y'''(t)] + 2\mathcal{L}[y''(t)] - \mathcal{L}[y'(t)] - 2\mathcal{L}[y(t)] = \mathcal{L}[0]$$

$$\{s^3\mathcal{L}[y(t)] - s^2y(0) - sy'(0) - y''(0)\} +$$
$$2\{s^2\mathcal{L}[y(t)] - sy(0) - y'(0)\} - \{s\mathcal{L}[y(t)] - y(0)\}$$
$$-2\mathcal{L}[y(t)] = 0$$

$$\mathcal{L}[y(t)]\{s^3 + 2s^2 - s - 2\} - s^2y(0) - sy'(0) - y''(0)$$
$$-2sy(0) - 2y'(0) + y(0) = 0$$

$$\mathcal{L}[y(t)]\{s^2(s+2) - 1(s+2)\} - s^2(0) - s(0) - 6 - 2(0)$$
$$-2(0) + 0 = 0$$

$$\mathcal{L}[y(t)]\{(s+2)(s^2-1)\} - 0 - 0 - 6 - 0 - 0 = 0$$

$$\mathcal{L}[y(t)]\{(s+2)(s^2-1)\} - 6 = 0$$

$a^2 - b^2 = (a+b)(a-b)$

$$\mathcal{L}[y(t)]\{(s+2)(s+1)(s-1)\} = 6$$

$$\mathcal{L}[y(t)] = \frac{6}{(s+2)(s-1)(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{6}{(s+2)(s-1)(s+1)}\right] \text{ — } ⊛$$

By partial fractions we have to solve

$$\frac{6}{(s+2)(s-1)(s+1)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{s+1} \text{ — ①}$$

x^{ly} B.S by $(s+2)(s-1)(s+1)$

$6 = A(s-1)(s+1) + B(s+2)(s+1) + C(s+2)(s-1)$

put $s=-2$

$6 = A(-3)(-1) + B(0) + C(0)$

$6 = 3A \Rightarrow \underline{\underline{A=2}}$

put $s=1$

$6 = A(0) + B(3)(2) + C(0)$

$6 = 0 + 6B + 0 \Rightarrow 6B = 6 \Rightarrow B = 1$

put $s=-1$

$6 = A(-2)(0) + B(0) + C(1)(-2)$

$6 = 0 + 0 - 2C$

$-2C = 6 \Rightarrow \underline{C=-3}$

put A, B, C in ①

$$\frac{6}{(s+2)(s-1)(s+1)} = \frac{2}{s+2} + \frac{1}{s-1} + \frac{-3}{s+1}$$

$$L^{-1}\left[\frac{6}{(s+2)(s-1)(s+1)}\right] = 2L^{-1}\left[\frac{1}{s+2}\right] + L^{-1}\left[\frac{1}{s-1}\right] - 3L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}\left[\frac{6}{(s+2)(s-1)(s+1)}\right] = 2e^{-2t} + e^{t} - 3e^{-t}$$

(*) becomes

$$\underline{\underline{y(t) = 2e^{-2t} + e^{t} - 3e^{-t}}}$$

$\cancel{L^{-1}\{f(s)\} = f(t)}$

③ Solve the following initial value problem by using Laplace transform

June 2016

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}; \ y(0) = 0, y'(0) = 0$$

Soln: Given $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$

$y(0) = 0, \ y'(0) = 0$

$y''(t) + 4y'(t) + 4y(t) = e^{-t}$

Taking Laplace transform on both sides

$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L[e^{-t}]$

$\{s^2 L[y(t)] - sy(0) - y'(0)\} + 4\{sL[y(t)] - y(0)\}$

$+ 4L[y(t)] = \frac{1}{s+1}$

Using the given initial conditions we obtain

$L[y(t)]\{s^2 + 4s + 4\} - sy(0) - y'(0) - 4y(0)$

$= \frac{1}{s+1}$

$L[y(t)]\{s^2 + 4s + 4\} - s(0) - 0 - 4(0) = \frac{1}{s+1}$

$L[y(t)]\{s^2 + 4s + 4\} - 0 - 0 - 0 = \frac{1}{s+1}$

$L[y(t)]\{(s+2)^2\} = \frac{1}{s+1}$

$L[y(t)] = \frac{1}{(s+1)(s+2)^2}$

$y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$ — (*)

$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}$ — ①

×ly B.S by $(s+1)(s+2)^2$

$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$ — ②

put $s = -1$

$1 = A(-1+2) + B(0) + C(0)$

$1 = A(1) \Rightarrow A = 1$

put $s = -2$

$1 = A(0) + B(0) + C(-1)$

$1 = 0 + 0 - C$

$C = -1$

equating the co-efficient of s^2 on B.S of ②

$0 = A + B$

w.k.T $A = 1$

$B = -1$

$1 = A(s^2+4+4s) + B(s^2+3s+2) + C(s+1)$

In LHS there is no s^2

so co-eff of s^2 is zero

$0 = A + B$

$A = 1$

$B = -1$

hence

① $\Rightarrow$

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= e^{-t} - e^{-2t} - e^{-2t} \cdot \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \quad (\because a = -2)$$

$$= e^{-t} - e^{-2t} - e^{-2t} \cdot t$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = e^{-t} - e^{-2t}(1+t)$$

(*) becomes

$$y(t) = e^{-t} - e^{-2t}(1+t)$$

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(ii) Employ laplace transform to solve the equation $y''+5y'+6y=5e^{2x}$, $y(0)=2$, $y'(0)=1$

Sol^n: Given $y''(x)+5y'(x)+6y(x)=5e^{2x}$ — (1)

$y(0)=2,\ y'(0)=1$

Taking laplace transform on both sides of (1)

$$\mathcal{L}[y''(x)]+5\mathcal{L}[y'(x)]+6\mathcal{L}[y(x)]=5\mathcal{L}[e^{2x}]$$

$$\{s^2\mathcal{L}[y(x)]-sy(0)-y'(0)\}+5\{s\mathcal{L}[y(x)]-y(0)\}+6\mathcal{L}[y(x)]=\frac{5}{s-2}$$

$$\mathcal{L}[y(x)]\{s^2+5s+6\}-sy(0)-y'(0)-5y(0)=\frac{5}{s-2}$$

Use the initial condition

$$\mathcal{L}[y(x)]\{s^2+5s+6\}-s(2)-1-5(2)=\frac{5}{s-2}$$

$$\mathcal{L}[y(x)]\{\underbrace{s^2+5s+6}\}-2s-11=\frac{5}{s-2}$$

$$\mathcal{L}[y(x)]\{(s+2)(s+3)\}-(2s+11)=\frac{5}{s-2}$$

$$\mathcal{L}[y(x)]\{(s+2)(s+3)\}=\frac{5}{s-2}+(2s+11)$$

$$\mathcal{L}[y(x)]\{(s+2)(s+3)\}=\frac{5+(2s+11)(s-2)}{s-2}$$

$$\mathcal{L}[y(x)]=\frac{5+2s^2+11s-4s-22}{(s+2)(s+3)(s-2)}$$

$$\mathcal{L}[y(x)]=\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}$$

$$y(x)=\mathcal{L}^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right] \quad \text{— (*)}$$

④ Employ Laplace transform to solve the equation $y''+5y'+6y=5e^{2x}$, $y(0)=2$, $y'(0)=1$

Sol^n: Given $y''(x)+5y'(x)+6y(x)=5e^{2x}$ — ①

$y(0)=2$, $y'(0)=1$

Taking Laplace transform on both sides of ①

$$L[y''(x)]+5L[y'(x)]+6L[y(x)]=5L[e^{2x}]$$

$$\{s^2L[y(x)]-sy(0)-y'(0)\}+5\{sL[y(x)]-y(0)\}+6L[y(x)]=\frac{5}{s-2}$$

$$L[y(x)]\{s^2+5s+6\}-sy(0)-y'(0)-5y(0)=\frac{5}{s-2}$$

Use the initial condition

$$L[y(x)]\{s^2+5s+6\}-s(2)-1-5(2)=\frac{5}{s-2}$$

$$L[y(x)]\{s^2+5s+6\}-2s-11=\frac{5}{s-2}$$

$$L[y(x)]\{(s+2)(s+3)\}-(2s+11)=\frac{5}{s-2}$$

$$L[y(x)]\{(s+2)(s+3)\}=\frac{5}{s-2}+(2s+11)$$

$$L[y(x)]\{(s+2)(s+3)\}=\frac{5+(2s+11)(s-2)}{s-2}$$

$$L[y(x)]=\frac{5+2s^2+11s-4s-22}{(s+2)(s+3)(s-2)}$$

$$L[y(x)]=\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}$$

$$y(x)=L^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right] \quad \text{—} \; * \qquad -②$$

$$\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3} \quad \text{--- (1)}$$

$\times^{ly}$ B.S by $(s-2)(s+2)(s+3)$

$$2s^2+7s-17 = A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)$$

put $s=2$

$$2(2)^2+7(2)-17 = A(4)(5) + B(0) + C(0)$$

$$8+14-17 = 20A + 0 + 0$$

$$5 = 20A \Rightarrow \underline{\underline{A = 1/4}}$$

put $s=-2$

$$2(-2)^2+7(-2)-17 = A(0) + B(-2-2)(-2+3) + C(0)$$

$$2(4)-14-17 = 0 + B(-4)(1) + 0$$

$$+23 = +4B$$

$$\underline{\underline{B = 23/4}}$$

put $s=-3$

$$2(-3)^2+7(-3)-17 = A(0) + B(0) + C(-3-2)(-3+2)$$

$$2(9)-21-17 = 0 + 0 + C(-5)(-1)$$

$$18-21-17 = 5C$$

$$-20 = 5C \Rightarrow C = \frac{-20}{5}$$

$$\underline{\underline{C = -4}}$$

(1) $\Rightarrow$

$$\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)} = \frac{1}{4(s-2)} + \frac{23}{4(s+2)} + \frac{-4}{(s+3)}$$

$$\mathcal{L}^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{23}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$$

$$\mathcal{L}^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right] = \frac{1}{4}e^{2x} + \frac{23}{4}e^{-2x} - 4e^{-3x}$$

(*) becomes

$$y(x) = \frac{1}{4}e^{2x} + \frac{23}{4}e^{-2x} - 4e^{-3x}$$

(5) Using laplace tranform technique solve $x'' - 2x' + x = e^{2t}$ with $x(0) = 0,\ x'(0) = -1$

June 2017

Soln: Given $x'' - 2x' + x = e^{2t}$

$$x''(t) - 2x'(t) + x(t) = e^{2t}$$

Take laplace tranform on B.S

$$\mathcal{L}[x''(t)] - 2\mathcal{L}[x'(t)] + \mathcal{L}[x(t)] = \mathcal{L}[e^{2t}]$$

$$\{s^2\mathcal{L}[x(t)] - sx(0) - x'(0)\} - 2\{s\mathcal{L}[x(t)] - x(0)\} + \mathcal{L}[x(t)] = \frac{1}{s-2}$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} - sx(0) - x'(0) - 2x(0) = \frac{1}{s-2}$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} - s(0) - (-1) - 2(0) = \frac{1}{s-2}$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} - 0 + 1 - 0 = \frac{1}{s-2}$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} + 1 = \frac{1}{s-2}$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} = \frac{1}{s-2} - 1$$

$$L[x(t)]\{s^2-2s+1\} = \frac{1-s+2}{s-2}$$

$$L[x(t)] = \frac{3-s}{(s-2)(s^2-2s+1)}$$

$$L[x(t)] = \frac{3-s}{(s-2)(s-1)^2}$$

$$x(t) = L^{-1}\left[\frac{3-s}{(s-2)(s-1)^2}\right] \quad \text{--- } *$$

$$\frac{3-s}{(s-2)(s-1)^2} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \quad \text{--- } (1)$$

$\times^{ly}$ $(s-2)(s-1)^2$

$$3-s = A(s-1)^2 + B(s-1)(s-2) + C(s-2) \quad \text{--- } (2)$$

put $s=2$

$$3-2 = A(2-1)^2 + B(0) + C(0)$$

$$1 = A(1) + 0 + 0 \Rightarrow A = 1$$

put $s=1$

$$3-1 = A(0) + B(0) + C(-1)$$

$$2 = -C \Rightarrow \underline{C=-2}$$

equating the co-efficient of s^2 on both sides of (2)

There is no s^2 in the LHS so

$$0 = A(s^2-2s+1) + B(s^2-3s+2) + C(s-2)$$

$$0 = A + B$$

$$0 = A + B$$

w.k.t $A = 1$

$$0 = 1 + B \Rightarrow \underline{B=-1}$$

put A, B, C values in (1)

$$\frac{3-s}{(s-2)(s-1)^2} = \frac{1}{s-2} - \frac{1}{s-1} - \frac{2}{(s-1)^2}$$

$$L^{-1}\left[\frac{3-s}{(s-2)(s-1)^2}\right] = L^{-1}\left[\frac{1}{s-2}\right] - L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$= e^{2t} - e^{t} - 2e^{t} \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$

$= e^{2t} - e^{t} - 2e^{t} t$

$\mathcal{L}^{-1}\left[\frac{3-5}{(s-2)(s-1)^2}\right] = e^{2t} - e^{t}(1+2t)$

(*) $\Rightarrow x(t) = e^{2t} - e^{t}(1+2t)$

Doyourself

June 2015 (6) Solve by using Laplace transform $\mathcal{L}$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-t} \text{ with } y(0) = 1, y'(0) = -2$$

(7) Solve $y'' + 6y' + 9y = 12t^2e^{-3t}$ subject to the conditions $y(0) = 0 = y'(0)$ by using the Laplace transforms.

Dec 2017

Soln. the given equation is

$$y''(t) + 6y'(t) + 9y(t) = 12t^2e^{-3t}$$

Take Laplace transform on B.S

$$\mathcal{L}[y''(t)] + 6\mathcal{L}[y'(t)] + 9\mathcal{L}[y(t)] = 12\mathcal{L}[t^2e^{-3t}]$$

$$\{s^2\mathcal{L}[y(t)] - sy(0) - y'(0)\} + 6\{s\mathcal{L}[y(t)] - y(0)\}$$

$$+ 9\mathcal{L}[y(t)] = 12\cdot\mathcal{L}[t^2]_{s\to s+3}$$

$$\mathcal{L}[y(t)]\{s^2+6s+9\} - sy(0) - y'(0) - 6y(0) = 12\left[\frac{2!}{s^3}\right]_{s\to s+3}$$

$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$$\mathcal{L}[y(t)]\{s^2+6s+9\} - 0 - 0 - 0 = \frac{24}{(s+3)^3}$$

$$\mathcal{L}[y(t)]\{(s+3)^2\} = \frac{24}{(s+3)^3}$$

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$$\mathcal{L}[y(t)] = \frac{24}{(s+3)^3(s+3)^2}$$

$$\mathcal{L}[y(t)] = \frac{24}{(s+3)^5}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{24}{(s+3)^5}\right]$$

$$= 24\,\mathcal{L}^{-1}\left[\frac{1}{(s+3)^5}\right]$$

$$= 24\, e^{-3t}\,\mathcal{L}^{-1}\left[\frac{1}{s^5}\right]$$

$$= 24 e^{-3t} \cdot \frac{t^4}{4!}$$

$$= \cancel{24} \cdot e^{-3t} \cdot \frac{t^4}{\cancel{24}}$$

$$\boxed{y(t) = e^{-3t} \cdot t^4}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^5}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{4+1}}\right] = \frac{t^4}{4!}$$

$(\because n=4)$

(8) Solve the following boundary value problem using Laplace transforms

$y''(t) + y(t) = 0;\ y(0) = 2,\ y(\pi/2) = 1$

Solⁿ: $y''(t) + y(t) = 0$

Take Laplace transform on both sides

$$\mathcal{L}[y''(t)] + \mathcal{L}[y(t)] = \mathcal{L}[0]$$

$$\mathcal{L}[y''(t)] + \mathcal{L}[y(t)] = 0 \quad \text{— (1)}$$

$$\{s^2\mathcal{L}[y(t)] - s\,y(0) - y'(0)\} + \mathcal{L}[y(t)]$$

Let us assume $y'(0) = a$, where a is a constant to be found later & we have $y(0) = 2$ by data

hence eqⁿ (1) becomes

$$(s^2+1)\mathcal{L}[y(t)] - 2s - a = 0$$

$$(s^2+1)\mathcal{L}[y(t)] = 2s+a$$

$$\mathcal{L}[y(t)] = \frac{2s+a}{s^2+1}$$

$$\mathcal{L}[y(t)] = \frac{2s+a}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s+a}{s^2+1}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2s}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{a}{s^2+1}\right]$$

$$= 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + a\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$y(t) = 2\cos t + a\sin t \quad - \text{(*)}$$

now we shall use the condition $y(\pi/2) = 1$

$$y(\pi/2) = 2\cos\pi/2 + a\sin\pi/2$$

$$1 = 2(0) + a(1)$$

$$1 = 0 + a$$

$$\underline{\underline{a = 1}}$$

$$\text{(*)} \Rightarrow \underline{\underline{y(t) = 2\cos t + \sin t}}$$

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yourself

Solve the DE $y''+4y'+3y=e^{-t}$ with $y(0)=1=y'(0)$ using laplace transforms.

ans: $y(t)=\frac{7}{4}e^{-t}+\frac{1}{2}e^{-t}\cdot t-\frac{3}{4}e^{-3t}$

by partial fractions we have to solve

Soln: Given $y''+4y'+3y=e^{-t}$

$y''(t)+4y'(t)+3y(t)=e^{-t}$

Taking laplace transform on both sides

$L[y''(t)]+4L[y'(t)]+3L[y(t)]=L[e^{-t}]$

$\{s^2L[y(t)]-sy(0)-y'(0)\}+4\{sL[y(t)]-y(0)\}$

$+3L[y(t)]=\frac{1}{s+1}$

$L[y(t)]\{s^2+4s+3\}-sy(0)-y'(0)-4y(0)$

$=\frac{1}{s+1}$

Use initial conditions

$L[y(t)]\{s^2+4s+3\}-s(1)-(1)-4(1)=\frac{1}{s+1}$

$L[y(t)]\{s^2+3s+s+3\}-s-5=\frac{1}{s+1}$

$L[y(t)]\{(s+3)(s+1)\}-(s+5)=\frac{1}{s+1}$

$L[y(t)]\{(s+3)(s+1)\}=\frac{1}{s+1}+s+5$

$L[y(t)]\{(s+3)(s+1)\}=\frac{1+(s+5)(s+1)}{(s+1)}$

$L[y(t)]\{(s+3)(s+1)\}=\frac{1+s^2+s+5s+5}{(s+1)}$

$L[y(t)]\{(s+3)(s+1)\}=\frac{s^2+6s+6}{(s+1)}$

$$L[y(t)] = \frac{s^2+6s+6}{(s+1)(s+1)(s+3)}$$

$$L[y(t)] = \frac{s^2+6s+6}{(s+1)^2(s+3)}$$

$$y(t) = L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right] \quad \text{---} \ (*)$$

by partial fractions we have to solve

$$\frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3} \quad \text{---} \ (1)$$

$\times^{ly}$ B.S by $(s+1)^2(s+3)$

$s^2+6s+6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2$ —

put $s = -1$

$(-1)^2 \cancel{-6} + \cancel{6} = A(0) + B(-1+3) + C(0)$

$1 = 0 + 2B + 0$

$2B = 1 \Rightarrow \underline{\underline{B = 1/2}}$

put $s = -3$

$(-3)^2 + 6(-3) + 6 = A(-3+1)(0) + B(0) + C(-3+1)^2$

$9 - 18 + 6 = 0 + 0 + C(-2)^2$

$-3 = 4C$

$\underline{\underline{C = -3/4}}$

equaring the co-efficient of s^2 on B.S of (2)

$1 = A + C$

$1 = A - 3/4 \Rightarrow A = 1 + 3/4$

$\underline{\underline{A = 7/4}}$

A, B, C values in ①

$$\frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{7}{4(s+1)} + \frac{1}{2(s+1)^2} - \frac{3}{4(s+3)}$$

$$L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right] = \frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}L^{-1}\left[\frac{1}{s^2}\right] - \frac{3}{4}e^{-3t}$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}\, t - \frac{3}{4}e^{-3t}$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}$$

∴ ⊛ becomes

$$y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}\, t - \frac{3}{4}e^{-3t}$$

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(6) $y''(t) + 2y'(t) + y(t) = te^{-t}$

$$L[y''(t)] + 2L[y'(t)] + L[y(t)] = L[te^{-t}]$$

$$\{s^2L[y(t)] - sy(0) - y'(0)\} + 2\{sL[y(t)] - y(0)\}$$
$$+ L[y(t)] = L[t]_{s\to s+1}$$

$$L[y(t)]\{s^2+2s+1\} - sy(0) - y'(0) - 2y(0)$$
$$= \left[\frac{1!}{s^2}\right]_{s\to s+1}$$

$$L[y(t)]\{(s+1)^2\} - s(1) - \cancel{(-2)} - \cancel{2(1)} = \frac{1}{(s+1)^2}$$

$$L[y(t)]\{(s+1)^2\} - s = \frac{1}{(s+1)^2}$$

$$L[y(t)]((s+1)^2) = \frac{1}{(s+1)^2} + s$$

$$L\,\cancel{[y(t)](s+1)^2} = \cancel{\frac{1+s(s+1)^2}{(s+1)^2}}$$

$$L\,\cancel{[y(t)]} = \cancel{\frac{1+s(s^2+1+2s)}{(s+1)^2(s+1)^2}} = \frac{1+s^3+s+2s^2}{(s+1)^4}$$

$$L[y(t)] = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}$$

$$y(t) = L^{-1}\left[\frac{1}{(s+1)^4}\right] + L^{-1}\left[\frac{s}{(s+1)^2}\right]$$

$$= e^{-t} L^{-1}\left[\frac{1}{s^4}\right] + L^{-1}\left[\frac{(s+1)-1}{(s+1)^2}\right]$$

$$= e^{-t} \cdot \frac{t^3}{3!} + L^{-1}\left[\frac{s-1}{s^2}\right]$$

$$= e^{-t} \cdot \frac{t^3}{6} + L^{-1}\left[\frac{\cancel{s}}{s^{\cancel{2}}}\right] - L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-t} \cdot \frac{t^3}{6} + L^{-1}\left[\frac{1}{s}\right] - t$$

$$= e^{-t} \cdot \frac{t^3}{6} + (1) - t$$

$$y(t) = e^{-t} \cdot \frac{t^3}{6} + 1 - t$$

List of formulas

① $\frac{d}{dx}(K) = 0$

② $\frac{d}{dx}(x^n) = nx^{n-1}$

③ $\frac{d}{dx}(e^x) = e^x$

④ $\frac{d}{dx}(e^{ax}) = ae^{ax}$

⑤ $\frac{d}{dx}(\log x) = \frac{1}{x}$

⑥ $\frac{d}{dx}(1/x) = -1/x^2$

⑦ $\frac{d}{dx}(1/x^2) = -2/x^3$

⑧ $\frac{d}{dx}(\sin x) = \cos x$

⑥ $\frac{d}{dx}(\cos x) = -\sin x$

⑦ $\frac{d}{dx}(\tan x) = -\sin x$

⑧ $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

⑨ $\frac{d}{dx}(\sec x) = \sec x \tan x$

⑩ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

⑪ $\frac{d}{dx}(\sin ax) = a\cos ax$

⑫ $\frac{d}{dx}(\cos ax) = -a\sin ax$

⑬ $\frac{d}{dx}(\tan ax) = a\sec^2 ax$

⑭ $\frac{d}{dx}(\cot ax) = -a\operatorname{cosec}^2 ax$

⑮ $\frac{d}{dx}(\operatorname{cosec} ax)$
$= -a\operatorname{cosec} ax \cot ax$

⑯ $\frac{d}{dx}(\sec ax) = a\sec ax \tan ax$

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① $\int K\,dx = Kx + C$

② $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

③ $\int e^x\,dx = e^x + C$

④ $\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$

⑤ $\int \frac{1}{x}\,dx = \log_e x + C$

⑥ $\int 1\cdot dx = x + C$

⑦ $\int \sin x\,dx = -\cos x + C$

⑧ $\int \cos x\,dx = \sin x + C$

⑨ $\int \tan x\,dx = -\log|\cos x| + C$ or $\log|\sec x| + C$

⑩ $\int \cot x\,dx = \log|\sin x| + C$

⑪ $\int \sec x\,dx = \log|\sec x + \tan x| + C$

⑫ $\int \sec x \tan x\,dx = \sec x + C$

⑬ $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$

⑭ $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$

⑮ $\int \sin ax\,dx = \frac{-\cos ax}{a} + C$

⑯ $\int \cos ax\,dx = \frac{\sin ax}{a} + C$

⑰ $\int \tan ax\,dx = \frac{-\log|\cos ax|}{a} + C$

(17) $\int \cot ax \, dx = \frac{\log|\sin ax|}{a} + C$

(18) $\int \operatorname{cosec} ax \, dx = \frac{\log|\operatorname{cosec} ax - \cot ax|}{a} + C$

(19) $\int \sec ax \, dx = \frac{\log|\sec ax + \tan ax|}{a} + C$

(20) $\int \sec^2 x \, dx = \tan x + C$

(21) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

Product rule & quotient rule of Differentiation

(1) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(2) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Bernoulli's Rule

$\int uv \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$

where u', u'', u''', u^{IV} are succesive differentiation
v_1, v_2, v_3, v_4 are succesive integrals of v.

where $v_1 = \int v \, dx \quad v_2 = \int v_1 \, dx$

eg: (1) $\int x^3 e^x \, dx = x^3 e^x - (3x^2)e^x + (6x)e^x - (6)e^x$

(2) $\int (x+x^2)\cos nx \, dx = (x+x^2)\left(\frac{\sin nx}{n}\right)$

$+ (1+2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)$

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NOTE: $\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$\cos 2n\pi = 1$

$\cos(2n+1)\pi = -1$

$\cos\pi = -1$, * $\cos 3\pi = \cos 5\pi = \cos 7\pi = \dots = -1$

* $\cos 2\pi = \cos 4\pi = \cos 6\pi = 1$

NOTE: $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Module 2

Engineering mathematics - III

Fourier Series

Definition of periodic function

A real valued function $f(x)$ is said to be periodic of period T. If $f(x+T)=f(x)$, $T>0$

eg: $\sin(x+2\pi) = \sin x$

$\cos(x+2\pi) = \cos x$

$f(x) = K$ then $f(x+2K) = f(x) = K$

$\therefore f(x+2K) = K$

Trigonometric Series and Eulers formula

The function K, $\cos nx$, $\sin nx$ (for $n = 1, 2, 3 \dots$) are all periodic functions of period 2π.

Then $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are all constants is called a trigonometric series

Euler's is defined the constants a_0, a_n and b_n

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Problems

① obtain the fourier series of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$ hence deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

Soln: The fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -①$$

Now we have to find a_0, a_n, b_n

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \frac{(2\pi)^2}{2}\right]$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \frac{4\pi^2}{2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2 - 4\pi^2}{2}\right]$$

$$= \frac{1}{2\pi} \times 0$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\cos nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos nx \, dx$$

Apply Bernoulli's rule

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (0-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{(\pi - 2\pi)\frac{\cancel{\sin(2n\pi)}^{\,0}}{n} - \frac{\cos(2n\pi)}{n^2}\right\} - \left\{(\pi - 0)\frac{\sin(0)}{n}\,\Big\downarrow_0 - \frac{\cos(0)}{n^2}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2n\pi}{n^2} + \frac{\cos(0)}{n^2}\right]$$

$\cos 2n\pi = 1$

$$= \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{2\pi} \times 0$$

$\underline{a_n = 0}$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2} \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x) \sin nx \, dx$$

Apply Bernoulli's rule

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (0-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - \cancel{\frac{\sin nx}{n^2}}^{\,0}\right]_0^{2\pi}$$

w.k.t $\sin 2n\pi = 0 = \sin 0$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(\pi - 2\pi)\left(-\frac{\cos 2n\pi}{n}\right) - (\pi - 0)\left(-\frac{\cos(0)}{n}\right)\right]$$

$$= \frac{1}{2\pi}\left[-\pi \times -\frac{\cos 2n\pi}{n} + \pi \frac{\cos(0)}{n}\right]$$

$$= \frac{1}{2\pi}\left[\pi \times \frac{1}{n} + \pi \times \frac{1}{n}\right]$$

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$$= \frac{1}{2\pi}\left(\frac{\pi}{n} + \frac{\pi}{n}\right)$$

$$= \frac{1}{2\pi}\left(2\frac{\pi}{n}\right)$$

$$b_n = 1/n$$

$\therefore$ equation ① becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

put a_0, a_n, b_n in ①

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} (0) \cos nx + \sum_{n=1}^{\infty} 1/n \cdot \sin nx$$

$$= 0 + 0 + \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad \text{—— ②}$$

w.k.T $f(x) = \frac{\pi - x}{2}$

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad \text{—— ③}$$

to get required series put

$x = \pi/2$ in eqn ③

$$\frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty} \frac{\sin n\pi/2}{n}$$

$$\frac{\pi}{4} = \sin \pi/2 + \frac{\sin 2(\pi/2)}{2} + \frac{\sin 3(\pi/2)}{3} + \frac{\sin 4(\pi/2)}{4} + \dots$$

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$$\frac{\pi}{4} = \sin\frac{\pi}{2} + 0 + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + 0 + \cdots\cdots$$

$$\frac{\pi}{4} = 1 + 0 + \frac{1}{3}(-1) + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

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(2) obtain the fourier series of the function x^2 in $-\pi < x < \pi$ hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

Soln: Fourier series $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Now we have to find a_0, a_n, b_n by Euler's formula

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - x - \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\cancel{\pi}} \times \frac{2\cancel{\pi}^3}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

Apply Bernoulli's rule

$$a_n = \frac{1}{\pi}\left[x^2\cancel{\frac{\sin nx}{n}}^{\,0} - (2x)\left(\frac{-\cos nx}{n^2}\right) + (+2)\left(\cancel{-\frac{\sin nx}{n^3}}^{\,0}\right)\right]_{-\pi}^{\pi}$$

w.k.t $\sin n\pi = 0$

$$= \frac{1}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{2\pi\cos n\pi}{n^2}\right\} - \left\{\frac{-2\pi\cos n(-\pi)}{n^2}\right\}\right]$$

$$= \frac{1}{\pi n^2}\left[2\pi\cos n\pi + 2\pi\cos n\pi\right]$$

$$a_n = \frac{1}{\pi n^2}\left[4\pi\cos n\pi\right]$$

$$a_n = \frac{4}{n^2}\cos n\pi$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\cancel{-\frac{\sin nx}{n^2}}^{\,0}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right\} - \left\{\frac{-(-\pi)^2\cos n(-\pi)}{n} + \frac{2\cos n(-\pi)}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{-\pi^2\cos n\pi}{n}} + \cancel{\frac{2\cos n\pi}{n^3}} + \cancel{\frac{\pi^2\cos n\pi}{n}} - \cancel{\frac{2\cos n\pi}{n^3}}\right]$$

$$b_n = \frac{1}{\pi}(0) \Rightarrow \boxed{b_n = 0}$$

put a_0, a_n, b_n in equation ①

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$f(x) = x^2 = \pi^2/3 + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

put $x = 0$ in eqn ② to get required series

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(0)$$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(1)$$

$$-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3} = 4\left[-\frac{1}{1^2} + \frac{1}{2^2} + \frac{(-1)}{3^2} + \frac{1}{4^2} + \frac{(-1)}{5^2} + \dots\right]$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \dots$$

$$\cancel{-}\frac{\pi^2}{12} = \cancel{-}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots\right]$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots \rightarrow (3)$$

put $x = \pi$ in eqn (2)

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos n\pi$$

w.k.t

$\cos n\pi = (-1)^n$

$(-1)^{2n} = 1$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n(-1)^n}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{\cancel{2}\pi^2}{\cancel{12}\,6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \rightarrow (4)$$

By adding equation (3) & (4)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2}\right) + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}\right)$$

$$\frac{3\pi^2}{12} = 2\cdot\frac{1}{1^2} + 2\cdot\frac{1}{3^2} + 2\cdot\frac{1}{5^2} + \cdots$$

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$$\frac{3\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{3\pi^2}{24} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

(3) If $f(x) = x(2\pi - x)$ in $0 \le x \le 2\pi$, show that $f(x) = \frac{2\pi^2}{3} - 4\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right)$

hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Solⁿ: The fourier series of $f(x)$ having period 2π is given by -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (1)}$$

we have to find a_0, a_n, b_n by Euler's formula.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{\cancel{2}\pi x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3} - 0\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{4\pi^{\cancel{3}}}{3}\right]$$

$$\boxed{a_0 = \frac{4\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\cos nx \, dx$$

Apply Bernoulli's rule

$$= \frac{1}{\pi}\left[\cancel{(2\pi x - x^2)\left(\frac{\sin nx}{n}\right)}^{0} - (2\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + \cancel{(0-2)\left(-\frac{\sin nx}{n^3}\right)}^{0} \right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[+ \frac{(2\pi - 2x)\cos nx}{n^2} \right]_0^{2\pi}$$

$$= +\frac{1}{\pi}\left[\frac{(2\pi - 2x)\cos nx}{n^2} \right]_0^{2\pi}$$

$$= +\frac{1}{\pi}\left[\left\{ \frac{2\pi - 2(2\pi)\cos 2n\pi}{n^2} \right\} - \left\{ \frac{(2\pi - 0)\cos(0)}{n^2} \right\} \right]$$

$$= \frac{+1}{n^2\pi}\left[-2\pi \cdot \cos 2n\pi - 2\pi(1) \right]$$

$\cos 2n\pi = 1$

$$= \frac{+1}{\pi n^2}\left[-2\pi(1) - 2\pi \right] = +\frac{1}{n^2\cancel{\pi}} \times -4\cancel{\pi} = \frac{-4}{n^2}$$

$$\therefore \boxed{a_n = \frac{-4}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - \cancel{(2\pi - 2x)}\left(\frac{-x\sin nx}{n^2}\right)^{\to 0} + (0-2)\left(\frac{-x-\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(2\pi x - x^2)\cos nx}{n} - \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-(\cancel{4\pi^2 - 4\pi^2})^{\to 0}\cos 2n\pi}{n} - \frac{2\cos 2n\pi}{n^3}\right\} - \left\{\frac{-2\cos(0)-0}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cancel{2}}{\cancel{n^3}}(1) + \frac{\cancel{2}}{\cancel{n^3}}(1)\right]$$

$$= \frac{1}{\pi}\times 0$$

$$\boxed{b_n = 0}$$

Then eqn ① becomes

$$f(x) = \frac{\cancel{4\pi^2}\,2\pi^2}{\cancel{3}\,2}\;(\text{i.e. } \tfrac{4\pi^2}{3}/2) + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx + 0$$

$$f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}\cos nx$$

$$f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

$$f(x) = \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right]$$

$$f(x) = x(2\pi - x)$$

$$x(2\pi - x) = \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right] \quad (*)$$

put $x = 0$

$$0 = \frac{2\pi^2}{3} - 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$-\frac{2\pi^2}{3} = -4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{2\pi^2}{3 \times 4_2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \rightarrow (3)$$

put $x = \pi$ in $(*)$

$$\pi(2\pi - \pi) = \frac{2\pi^2}{3} - 4\left[\frac{\cos\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \cdots\right]$$

$$\pi^2 - \frac{2\pi^2}{3} = -4\left[\frac{-1}{1^2} + \frac{1}{2^2} + \frac{(-1)}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{3} = -4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \rightarrow (4)$$

add eq^n (3) & eq^n (4)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\frac{2\pi^2+\pi^2}{12} = 2\times\frac{1}{1^2} + 2\times\frac{1}{3^2} + 2\times\frac{1}{5^2} + \cdots$$

$$\frac{\cancel{3}\pi^2}{\cancel{12}_{4}} = 2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

(*)

$$\frac{\pi^2}{4\times 2} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

NOTE: $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(4) Find the fourier series to represent e^{-ax} from $x=-\pi$ to $x=\pi$

Soln: The given $f(x)$ having period 2π the fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \quad \text{---} (1)$$

Now we have to find a_0, a_n & b_n by Euler's formula

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{-a\pi}\left[e^{-ax}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{-a\pi}\left[e^{-a\pi} - e^{-ax-\pi}\right]$$

$$= \frac{1}{-a\pi}\left[e^{-a\pi} - e^{a\pi}\right]$$

$$= \frac{1}{a\pi}\left[-e^{-a\pi} + e^{a\pi}\right]$$

$$= \frac{1}{\pi a}\left[e^{a\pi} - e^{-a\pi}\right]$$

x^{ly} & $\div$ by 2

$$= \frac{2}{\pi a}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]$$

W.K.T

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$$

$$\boxed{a_0 = \frac{2\sinh a\pi}{a\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}\left(-a\cos n\pi + \cancel{n\sin n\pi}^{\,0}\right) - \frac{e^{+a\pi}}{a^2+n^2}\left(-a\cos n(\pi) + \cancel{n\sin n(-\pi)}^{\,0}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{+a}{a^2+n^2}\left(-e^{-a\pi}\cos n\pi + e^{a\pi}\cos n\pi\right)\right]$$

$$= \frac{a\cos n\pi}{\pi(a^2+n^2)}\left[-e^{-a\pi} + e^{a\pi}\right]$$

x^{ly} & $\div$ by 2

$$= \frac{2a\cos n\pi}{\pi(a^2+n^2)}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]$$

W.K.T $\cos n\pi = (-1)^n$

$$a_n = \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}\left(\cancel{-a\sin nx}^{\,0} - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-a\pi}\cos n\pi - e^{+a\pi}\cos n(-\pi)\right]$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-a\pi}\cos n\pi - e^{a\pi}\cos n\pi\right]$$

$$= \frac{n}{\pi(a^2+n^2)}\left[e^{a\pi}\cos n\pi - e^{-a\pi}\cos n\pi\right]$$

$$= \frac{n\cos n\pi}{\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right]$$

$\times^{ly}$ & $\div$ by 2

$$= \frac{2n\cos n\pi}{\pi(a^2+n^2)}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]$$

$$b_n = \frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

w.k.T

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

put a_0, a_n & b_n value in ①

$$f(x) = \frac{\cancel{2}\sinh a\pi}{\frac{a\pi}{\cancel{2}}} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}\cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}\sin nx$$

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(5) obtain the fourier series for

$$f(x) = \begin{cases} -k & \text{in } (-\pi, 0) \\ k & \text{in } (0, \pi) \end{cases}$$ hence deduce

that $1 - 1/3 + 1/5 - 1/7 + \cdots$

Soln: ~~the fourier series of~~ $f(x)$ is defined in $(-\pi, \pi)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\int_{-\pi}^{0} 1\cdot dx + k\int_{0}^{\pi} 1\cdot dx\right]$$

$$= \frac{1}{\pi}\left[-k\,[x]_{-\pi}^{0} + k\,[x]_{0}^{\pi}\right]$$

$$= \frac{k}{\pi}\left[-[0-(-\pi)] + [\pi - 0]\right]$$

$$= \frac{k}{\pi}\left[-\cancel{(\pi)} + \cancel{\pi}\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_0^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -K\cos nx\,dx + \int_0^{\pi} K\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-K\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + K\left[\frac{\sin nx}{n}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}[0]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -K\sin nx\,dx + \int_0^{\pi} K\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-K\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + K\left[\frac{-\cos nx}{n}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-K\left[\frac{-\cos(0)}{n} - \left(\frac{-\cos n(-\pi)}{n}\right)\right] + K\left[\frac{-\cos n(\pi)}{n} - \left(\frac{-\cos(0)}{n}\right)\right]\right]$$

w.k.T $\cos(-\theta) = \cos\theta$

$\therefore \cos(-\pi) = \cos\pi$

$$= \frac{1}{\pi}\left[-K\left\{\frac{-1}{n} + \frac{\cos n(\pi)}{n}\right\} + K\left\{\frac{-\cos n\pi}{n} + \frac{1}{n}\right\}\right]$$

$$= \frac{K}{\pi}\left[-\left\{-\frac{1}{n} + \frac{\cos n\pi}{n}\right\} + \left\{\frac{-\cos n\pi}{n} + \frac{1}{n}\right\}\right]$$

$$= \frac{K}{\pi}\left[\frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{1}{n}\right]$$

$$= \frac{K}{\pi}\left[\frac{2}{n} - \frac{2\cos n\pi}{n}\right]$$

$$= \frac{2K}{\pi n}(1 - \cos n\pi)$$

w.k.T $\cos n\pi = (-1)^n$

$$b_n = \frac{2K}{\pi n}(1 - (-1)^n)$$

put a_0, a_n, b_n in equation ①

$$f(x) = \frac{0}{2} + 0 + \sum_{n=1}^{\infty} \frac{2K}{n\pi}(1-(-1)^n)\sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2K}{n\pi}(1-(-1)^n)\sin nx$$

$$1-(-1)^n = \begin{cases} 1-(-1)^1 = 1+1 = 2, & \text{where } n = 1, 3, 5, 7\ldots \quad n \text{ is odd} \\ 1-(-1)^2 = 1-1 = 0, & \text{where } n = 2, 4, 6\ldots \quad n \text{ is even} \end{cases}$$

$$f(x) = \frac{2K}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{2\sin nx}{n}$$

$$f(x) = \frac{4K}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{\sin nx}{n}$$

put $x = \pi/2$

$$f(x) = \frac{4K}{\pi}\sum_{n=1,3,5}^{\infty} \frac{\sin n(\pi/2)}{n}$$

$$f(x) = \frac{4K}{\pi}\left\{\frac{\sin\pi/2}{1} + \frac{\sin 2(\pi/2)}{2} + \frac{\sin 3(\pi/2)}{3} + \ldots\right\}$$

$$= \frac{4K}{\pi}\left\{\frac{1}{1} + 0 - \frac{1}{3} + 0 + \frac{1}{5} - \frac{1}{7}\right.$$

$f(x) = K$, since $f(x) = K$ in $0 < x < \pi$

$$\cancel{K} = \frac{4\cancel{K}}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

w.k.t $\sin\frac{\pi}{2} = 1$ $\sin\frac{3\pi}{2} = -1$

$\sin\frac{5\pi}{2} = 1$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

⑥ Find the Fourier Series of the function $f(x) = \begin{cases} -\pi & \text{in } -\pi < x < 0 \\ x & \text{in } 0 < x < \pi \end{cases}$

hence deduce that sum of reciprocal squares of odd integers is equal to $\pi^2/8$

Sol^n: $f(x)$ is defined in $(-\pi, \pi)$ and fourier series of $f(x)$, having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi . dx + \int_{0}^{\pi} x \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \int_{-\pi}^{0} 1 . dx + \int_{0}^{\pi} x \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \left[x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi \{0-(-\pi)) + (\pi^2/2 - 0)\right]$$

$$= \frac{1}{\pi}\left[-\pi (\pi) + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{-\pi^{\cancel{2}}}{2}\right]$$

$$\boxed{a_0 = \frac{-\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \cos nx \, dx + \int_{0}^{\pi} f(x) \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx \, dx + \int_{0}^{\pi} x \cos nx \, dx\right]$$

Apply Bernoulli's rule

$$= \frac{1}{\pi}\left[-\pi \int_{-\pi}^{0} \cos nx \, dx + \left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi\left[\frac{\cancel{\sin nx}}{n}\right]_{-\pi}^{0} + \left[\frac{(x\,\cancel{\sin nx})}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\cos nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi n^2}\left[\cos n\pi - \cos(0)\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

Taking minus common

$$\boxed{a_n = \frac{-1}{\pi n^2}\left[1 - (-1)^n\right]}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\, dx + \int_0^{\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\, dx + \int_0^{\pi} x\sin nx\, dx\right]$$

Apply Bernoulli's rule

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left(x\times\frac{-\cos nx}{n} - (1)\left(\frac{-1}{n}\times\frac{\sin nx}{n}\right)\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[+\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(\frac{-x\cos nx}{n} + \frac{\cancel{\sin nx}}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\frac{x\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos 0}{n} - \frac{\cos n(-\pi)}{n}\right) + \left(-\frac{\pi\cos n\pi}{n} + \underbrace{\frac{0\cos(0)}{n}}_{0}\right)\right]$$

$$= \frac{1}{\pi n}\left[\pi(1-\cos n\pi) - \pi\cos n\pi\right]$$

$$= \frac{\cancel{\pi}}{\cancel{\pi} n}(1-\cos n\pi - \cos n\pi)$$

$$= \frac{1}{n}(1-2\cos n\pi)$$

$$\boxed{b_n = \frac{1}{n}(1-2(-1)^n)}$$

put a_0, a_n, b_n in equation ①

$$f(x) = \frac{-\pi/2}{2} + \sum_{n=1}^{\infty}\frac{-1}{\pi n^2}\{1-(-1)^n\}\cos nx +$$

$$\sum_{n=1}^{\infty} \tfrac{1}{n}\{1-2(-1)^n\}\sin nx$$

To deduce the required series let us put $x=0$ in the fourier series. It should be observed from the given $f(x)$ that $x=0$ is a point of discontinuity & hence the series converges to

$$\frac{1}{2}\left[f(0^+)+f(0^-)\right] = \frac{1}{2}\left[0+(-\pi)\right]$$

$$= -\pi/2$$

Because to the right of 0, in $(0,\pi)$ $f(x)=x$ and $f(0^+)=0$. Also to the left of 0 in $(-\pi,0)$ $f(x)=-\pi$ and $f(0^-)=-\pi$

hence the fourier series becomes

$$-\frac{\pi}{2} = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{\pi n^2}\{1-(-1)^n\}(\cos 0)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n}\{1-2(-1)^n\} \underbrace{\sin(0)}_{0}$$

$$-\frac{\pi}{2} + \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{-1}{\pi n^2}\{1-(-1)^n\}$$

$$-\frac{\pi}{4} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}\{1-(-1)^n\}$$

$$\Rightarrow \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}\{1-(-1)^n\}$$

But $1-(-1)^n = \begin{cases} 1-(1) = 0 & \text{when } n \text{ is even} \\ 1-(-1) = 2 & \text{when } n \text{ is odd} \end{cases}$

$$\frac{\pi^2}{4} = \sum_{n=1,3,5\ldots}^{\infty} 2/n^2$$

$$\frac{\pi^2}{4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Do yourself

(8) If $f(x) = \begin{cases} x & \text{in } 0 \le x \le \pi \\ 2\pi - x & \text{in } \pi \le x \le 2\pi \end{cases}$

S.T fourier series of $f(x)$ in $[0, 2\pi]$ is $\frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right]$ and hence deduce that $\pi^2/8 = 1/1^2 + 1/3^2 + 1/5^2 + \cdots$

Ans: $a_0 = \pi$

$a_n = -\frac{2}{\pi n^2}\{1-(-1)^n\}$

$b_n = 0$

put $x=0$ to deduce given series.

Fourier series of even and odd functions in the interval $(-\pi, \pi)$

A function $f(x)$ is said to be even if $f(-x) = f(x)$

eg: $x^2, x^4, x^6 \ldots \cos x$ are even function.

$f(x) = x^2$

Replace x by $-x$

$f(-x) = (-x)^2 = x^2$

$f(-x) = f(x)$

even function

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

eg: $f(x) = x, x^3, x^5 \ldots, \sin x$ are odd functions.

$f(x) = x$

Replace x by $-x$

$f(-x) = -x$

$f(-x) = -f(x)$

odd function

Property ①: The product of two even functions and that of two odd function is always even. where as product of an even and an odd function is always odd.

Property ②: $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

Now, Suppose the periodic function $f(x)$ is defined in the interval $(-\pi, \pi)$ then fourier co-efficients are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Case ①: $f(x)$ is an even function then clearly to caluclate

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Case ②: $f(x)$ is odd function clearly to Caluclate $a_0 = 0$ $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

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Problems

① Find the fourier series of

$$f(x) = \begin{cases} 1+\dfrac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1-\dfrac{2x}{\pi} & \text{in } 0 < x < \pi \end{cases}$$

Sol: The fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -①$$

we shall check $f(x)$ is even or odd

$$f(x) = 1+\frac{2x}{\pi}$$

$$f(-x) = 1-\frac{2x}{\pi}$$

$$f(-x) = f(x)$$

(or) $f(x) = 1-\dfrac{2x}{\pi}$

$f(-x) = 1+\dfrac{2x}{\pi}$

$f(-x) = f(x)$

∴ Given function is even,

clearly $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{\cancel{2}x^2}{\cancel{2}\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

Apply Bernoulli's rule

$$= \frac{2}{\pi}\left[\pi - \frac{\pi^2}{\pi} - 0\right]$$

$$= \frac{2}{\pi}\left[\pi - \pi\right]$$

$$= \frac{2}{\pi} \times 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(0 - \frac{2}{\pi}\right)\left(\frac{1}{n}x - \frac{\cos nx}{n}\right)\right]_0^\pi$$

$$a_n = \frac{2}{\pi}\left[\cancel{\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)}^{0} - \frac{2}{\pi n^2}\cos nx\right]_0^\pi$$

$$a_n = \frac{2}{\pi}\left[-\frac{2}{\pi n^2}\cos nx\right]_0^\pi$$

$$a_n = \frac{2}{\pi}\left[-\frac{2}{\pi n^2}\left\{\cos n\pi - \cos 0\right\}\right]$$

$$a_n = \frac{-4}{\pi^2 n^2}\left\{(-1)^n - 1\right\}$$

take minus common

$$a_n = \frac{4}{\pi n^2}\left\{1 - (-1)^n\right\}$$

w.k.t $b_n = 0$

eqn ① becomes

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\left\{1 - (-1)^n\right\}\cos nx + 0$$

$$f(x) = \frac{4}{\pi n^2}\sum_{n=1}^{\infty}\left(1 - (-1)^n\right)\cos nx$$

period 2π i

$f(x) = \frac{a_0}{2} + \sum_{n=}$

we shall f

$f(x) = -K$

$f(-x) = -f($

(8) obtain the fourier series in $(-\pi, \pi)$ for $f(x) = x\cos x$

Soln: Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---(1)}$$

we shall chek $f(x) = x\cos x$ for even or odd nature

$f(x) = x\cos x$

Replace x by $-x$

$f(-x) = -x\cos(-x)$

$f(-x) = -x\cos x$ $\quad \{\cos(-\theta) = \cos\theta$

$f(-x) = -f(x)$

hence $f(x)$ is odd

clearly, $a_0 = 0 \quad a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx \cos x\, dx \rightarrow (2)$$

use $\sin A\cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$$= \frac{2}{\pi}\int_0^{\pi} x\left[\frac{1}{2}\{\sin(n+1)x + \sin(n-1)x\}\right]dx$$

$$= \frac{\cancel{2}}{\pi} \times \frac{1}{\cancel{2}}\left\{\int_0^{\pi} x\sin(n+1)x\, dx + \int_0^{\pi} x\sin(n-1)x\, dx\right\}$$

(5) Apply Bernoulli's rule

$$= \frac{1}{\pi}\left\{\left[x \times \frac{-\cos(n+1)x}{(n+1)} - (1) \times \frac{-1}{(n+1)} \times \frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi}\right.$$

$$\left. + \left[x \times \frac{-\cos(n-1)}{(n-1)}x - (1) \times \frac{-1}{(n-1)} \times \frac{\sin(n-1)x}{(n-1)}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x\cos(n+1)x}{(n+1)} + \underbrace{\frac{\sin(n+1)x}{(n+1)^2}}_{0}\right]_0^{\pi} + \left[\frac{-x\cos(n-1)x}{(n-1)} + \underbrace{\frac{\sin(n-1)x}{(n-1)^2}}_{0}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\pi\cos(n+1)\pi}{(n+1)} + 0\right] + \left[\frac{-\pi\cos(n-1)\pi}{(n-1)} + 0\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^{n+1}}{(n+1)} - \frac{\pi(-1)^{n-1}}{n-1}\right]$$

$$= \frac{-\cancel{\pi}}{\cancel{\pi}}\left[\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}\right]$$

$$= -\left[\frac{(-1)^n \cdot (-1)^1}{n+1} + \frac{(-1)^n \cdot (-1)^{-1}}{n-1}\right]$$

$$= -(-1)^n\left[\frac{-1}{n+1} + \frac{-1}{n-1}\right] \qquad \left\{(-1)^{-1} = \frac{1}{-1} = -1\right.$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{n-\cancel{1}+n+\cancel{1}}{n^2-1}\right]$$

$$b_n = \frac{2n(-1)^n}{n^2-1} \quad (n \neq 1)$$

eqn ① becomes

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2n(-1)^n}{n^2-1} \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2n(-1)^n \sin nx}{n^2-1} \longrightarrow ②$$

we shall now find b_n when $n=1$

i.e to find b_1

let us consider b_n as given by ②

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \cos x \, dx$$

put $n=1$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$\times^{ly}$ & $\div$ by 2

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \frac{2x \sin x \cos x}{2} \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x \sin 2x}{2} \, dx$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

Apply Bernoulli's Rule

$$b_1 = \frac{1}{\pi}\left[x \times \frac{-\cos 2x}{2} - (1) \times \frac{-1}{2} \times \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} \right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\pi \cos 2\pi - 0 \right]$$

$$= \frac{1}{2\cancel{\pi}} \times -\cancel{\pi} \times 1$$

$$b_1 = \frac{-1}{2}$$

hence eq^n ① can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$f(x) = 0 + 0 + \left(\frac{-1}{2}\right)\sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1} \sin nx$$

$$f(x) = \frac{-1}{2}\sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1} \sin nx$$

(3) Expand the function $f(x) = x\sin x$ as a fourier series in the interval $-\pi \le x \le \pi$ deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots = \frac{\pi - 2}{4}$$

Sol^n: Fourier series having period 2π is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

We shall check the nature of function

$$f(x) = x\sin x$$

$$f(-x) = -x\sin(-x)$$

$$= -x \times -\sin x$$

$$= x\sin x$$

$$f(-x) = f(x)$$

$\therefore$ $f(x)$ is even function

Consequently $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \quad a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

apply Bernoulli's rule

$$a_0 = \frac{2}{\pi}\left[x \times -\cos x - (1)\times -\underset{0}{\sin x}\right]_0^{\pi}$$

$$\sin\pi = 0 = \sin 0$$

$$= \frac{2}{\pi}\left[-x\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi + 0\right] \quad ; \quad a_0 = \frac{2}{\pi}(-\pi \times -1)$$

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$$a_0 = \frac{2\pi}{\pi}$$

$$a_0 = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx \longrightarrow (2)$$

using $\sin A\cos B = \frac{1}{2}\left\{\sin(A+B) + \sin(A-B)\right\}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cdot\frac{1}{2}\left\{\sin(x+nx) + \sin(x-nx)\right\}dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(1+n)x\,dx + \int_0^{\pi} x\sin(1-n)x\,dx\right.$$

$$\sin(1-n) = \sin(-(n-1)) = -\sin(n-1)$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(n+1)x\,dx + \int_0^{\pi} x\times -\sin(n-1)\,dx\right.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(n+1)x\,dx - \int_0^{\pi} x\sin(n-1)\,dx\right]$$

Apply Bernoulli's rule

$$= \frac{1}{\pi}\left\{\left[x\times\frac{-\cos(n+1)x}{(n+1)} - (1)\times\frac{-1}{n+1}\times\frac{\cancel{\sin(n+1)}}{n+1}\right]_0^{\pi}\right.$$

$$\left. - \left[x\times\frac{-\cos(n-1)x}{(n-1)} - (1)\times\frac{-1}{(n-1)}\times\frac{\cancel{\sin(n-1)x}}{(n-1)}\right]_0^{\pi}\right\}$$

But $\sin(n+1)\pi = 0 = \sin(n-1)\pi$

$$= \frac{1}{\pi}\left[\frac{-x\cos(n+1)x}{(n+1)} + \frac{x\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

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$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{(n+1)} + \frac{\pi\cos(n-1)\pi}{(n-1)} + 0 - 0\right]$$

$$= \frac{\cancel{\pi}}{\cancel{\pi}}\left[\frac{-\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)}\right]$$

$$= \frac{-1}{(n+1)} \times (-1)^{n+1} + \frac{1}{(n-1)} \times (-1)^{n-1}$$

$$= \frac{(-1)^1 \cdot (-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}$$

$$\left\{\begin{array}{l} \cos\pi = (-1) \\ \cos k\pi = (-1)^k \\ \cos n\pi = (-1)^n \end{array}\right.$$

$$= \frac{(-1)^{n+2}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}$$

$$= \frac{(-1)^n \cdot (-1)^2}{(n+1)} + \frac{(-1)^n \cdot (-1)^{-1}}{(n-1)}$$

$$= (-1)^n\left[\frac{(-1)^2}{n+1} + \frac{(-1)^{-1}}{n-1}\right]$$

$$\left\{\begin{array}{l} (-1)^2 = 1 \\ (-1)^{-1} = \frac{+1}{-1} = -1 \end{array}\right.$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{-1}{n-1}\right]$$

$$= (-1)^n\left[\frac{\cancel{n} - 1 - \cancel{n} - 1}{n^2 - 1}\right]$$

$$= (-1)^n \cdot \frac{(-2)}{n^2 - 1}$$

$$= \frac{2 \times -1 \times (-1)^n}{n^2 - 1}$$

$$= \frac{2(-1)^{n+1}}{n^2 - 1} \quad \text{where } n \neq 1$$

we shall now find a_n when $n=1$

i.e to find a_1

let us consider a_n as given by (2)

putting $n=1$ we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$$

x^{1y} & ÷ by 2

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x \times 2\sin x\cos x}{2}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x\sin 2x}{2}\,dx$$

$$= \frac{1}{\pi}\left[x \times \frac{-\cos 2x}{2} - (1)\times\frac{-\cancel{\sin 2x}}{2}\right]_0^{\pi}$$

$\sin 2\pi = 0$

$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{-\cancel{\pi}\cos 2\pi}{2} + 0\right]$$

$$= \frac{-\cos 2\pi}{2}$$

$$= \frac{-1}{2} \quad \text{w.k.T } \cos 2\pi = 1$$

$$a_1 = \frac{-1}{2}$$

put $b_n = 0$ in (1) we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{a_0}{2} + a_1\cos n x + \sum_{n=2}^{\infty} a_n\cos nx$$

put a_0, a_1, a_n

$$x\sin x = \frac{2}{2} + \frac{-1}{2}\times\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}\cos nx}{n^2-1}$$

To deduce the series, let us put $x = \pi/2$

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

since $\sin\frac{\pi}{2} = 1$ $\cos\frac{\pi}{2} = 0$

$$\frac{\pi}{2}\times 1 = 1 - 0 + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\frac{\pi}{2} - 1 = 2\left(\frac{(-1)^3}{3}\cos\frac{2\pi}{2} + \frac{(-1)^4}{8}\cos\frac{3\pi}{2} + \frac{(-1)^5}{15}\cos\frac{4\pi}{2} + \cdots\right)$$

$$\frac{\pi-2}{4} = \left(\frac{-1}{3}\times -1 + \frac{1}{8}\times 0 + \frac{(-1)(1)}{15} + \cdots\right)$$

$$\frac{\pi-2}{4} = \frac{1}{3} - \frac{1}{15} + \cdots$$

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

(4) Sketch the graph of the function $f(x) = |x|$ in $-\pi \le x \le \pi$ and obtain Fourier series. Hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

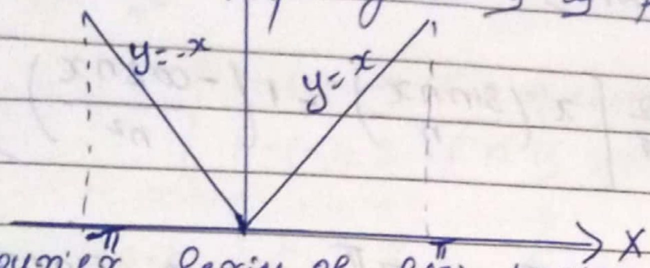
Soln: $f(x) = |x|$ in $-\pi \le x \le \pi$ means that the function the given interval which consists of negative values and hence given $f(x)$ may be split in to the form

$$f(x) = \begin{cases} -x & \text{in } -\pi \le x \le 0 \\ x & \text{in } 0 \le x \le \pi \end{cases}$$

The equations $y = x$ and $y = -x$ represents straight line through the origin

a_n

with slopes 1, -1 and graph y as follows



The fourier Series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -①$$

we shall check $f(x) = |x|$ for even or odd nature

$f(-x) = |-x| = |x| = f(x)$

$f(-x) = f(x)$ and hence $f(x)$ is even.

Consequently $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

here $f(x) = |x| = x$ for $x \in (0, \pi)$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$a_0 = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{\cancel{2}}{\pi} \times \frac{1}{\cancel{2}}\left[\pi^2 - 0\right]$$

$$= \frac{\pi^{\cancel{2}}}{\cancel{\pi}}$$

$$\underline{\underline{a_0 = \pi}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

Apply Bernoulli's rule

$$a_n = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[\cos nx\right]_0^{\pi}$$ since $\sin n\pi = 0 = \sin 0$

$$= \frac{2}{\pi n^2}(\cos n\pi - \cos 0)$$

$$= \frac{2}{\pi n^2}((-1)^n - 1)$$

take - common

$$a_n = \frac{-2}{\pi n^2}(1-(-1)^n)$$

put a_0, a_n, b_n in ①

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{\pi n^2}\left\{1-(-1)^n\right\}\cos nx$$

To deduce the series let us put $x=0$ in the fourier series

$$f(0) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-(-1)^n\right\}.\cos(0)$$

$$0 = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}(1-(-1)^n)(1)$$

$$-\frac{\pi}{2} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^2}$$

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}$$

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}(1-(-1)^n)$$

$$(1-(-1)^n) = \begin{cases} 1-(1) = 0 & \text{if } n \text{ is even} \\ 1-(-1) = 2 & \text{if } n \text{ is odd} \end{cases}$$

hence we get, $\frac{\pi^2}{4} = \sum_{n=1,3,5\ldots}^{\infty} \frac{2 \times 1}{n^2}$

$$\frac{\pi^2}{4} = 2 \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1,3,5\ldots}^{\infty} 1/n^2$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

obtain the fourier series for ~~f(x)=sinmx~~ $f(x) = \sin mx$ in the range $(-\pi, \pi)$ where m is neither zero nor an integer

Sol: fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

we shall check $f(x) = \sin mx$ for even or odd

$$f(-x) = \sin m(-x)$$
$$f(-x) = -\sin mx$$
$$f(-x) = -f(x)$$

$f(x)$ is an odd function

$a_0 = 0 \quad a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin mx \sin nx\,dx$$

$$\sin A \sin B = \frac{1}{2}\left\{\cos(A-B) - \cos(A+B)\right\}$$

$$b_n = \frac{\cancel{2}}{\pi}\int_0^{\pi}\frac{1}{\cancel{2}}\left\{\cos(m-n)x - \cos(m+n)x\right\}dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(m-n)\pi}{m-n} - \underbrace{\cancel{\frac{\sin(0)}{m-n}}}_{0}\right\} - \left\{\frac{\sin(m+n)\pi}{m+n} - \underbrace{\cancel{\frac{\sin(m+n)0}{m+n}}}_{0}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{m-n}(\sin(m\pi - n\pi)) - \frac{1}{m+n}\sin(m\pi + n\pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{m-n}\left[\sin m\pi\cos n\pi - \underbrace{\cancel{\cos m\pi \sin n\pi}}_{0}\right] - \frac{1}{m+n}\left[\sin m\pi \cos n\pi + \underbrace{\cancel{\cos m\pi \sin n\pi}}_{0}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{m-n}\left\{\sin m\pi\,(-1)^n\right\} - \frac{1}{m+n}\left\{\sin m\pi\,(-1)^n\right\}\right]$$

$$= \frac{1}{\pi}\left[\sin m\pi\,(-1)^n\left\{\frac{1}{m-n} - \frac{1}{m+n}\right\}\right]$$

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$$= \frac{1}{\pi}\left[\sin m\pi\,(-1)^n \left(\frac{\cancel{m}+\cancel{n}-\cancel{m}+n}{m^2-n^2}\right)\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\sin m\pi\,(-1)^n.\,(2n)}{m^2-n^2}\right]$$

put a_0, a_n, b_n in ①

$$f(x) = \sum_{n=1}^{\infty} \frac{2n\,(-1)^n \sin m\pi}{\pi(m^2-n^2)} \sin nx$$

Even and odd nature of $f(x)$ in $(0, 2\pi)$ and associated with fourier series

$f(x)$ is said to be even if $f(2\pi - x) = f(x)$

eg: $\cos(2\pi - x) = \cos x = f(x)$

$\therefore f(x)$ is even function

$f(x)$ is said to be odd if $f(2\pi - x) = -f(x)$

eg: $\sin(2\pi - x) = -\sin x = -f(x)$, odd

$\therefore f(x)$ is odd function

W.K.T Integral property $\int_0^{2a} f(x)\,dx =$

$$\begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

If $f(x)$ is a periodic function of period 2π defined in $(0, 2\pi)$ then fourier co-efficients are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Case ①: If $f(x)$ is even function

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Case ②: If $f(x)$ is odd function

$$a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{2}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

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① obtain the fourier Series expansion of the function

$$f(x) = \begin{cases} x & \text{in } 0 < x < \pi \\ x - 2\pi & \text{in } \pi < x < 2\pi \end{cases}$$

hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$

Ans: The fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— ①}$$

we shall check the given function is even or odd

$f(x) = x$

$f(2\pi - x) = 2\pi - x$

$= -(x - 2\pi)$

$= -f(x)$

$f(x) = -f(x)$

∴ Given function is odd

$a_0 = 0, \; a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi} \left[x \times \frac{-\cos nx}{n} - (1) \times \frac{-1}{n} \times \frac{\sin nx}{n} \right]_0^{\pi}$$

∵ $\sin n\pi = 0 = \sin 0$

$$= \frac{2}{\pi} \left[-\frac{x \cos nx}{n} \right]_0^{\pi}$$

$$= \frac{2}{\pi n} \left[-\pi \cos n\pi + 0 \right]$$

$$= \frac{2}{\pi n} \left[-\pi (-1)^n \right]$$

$$= \frac{2}{\cancel{\pi} n}\left(-\cancel{\pi}(-1)^n\right)$$

$$b_n = \frac{-2(-1)^n}{n}$$

put a_0, a_n, b_n in eqⁿ ①

$$f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$$f(x) = x$$

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

put $x = \pi/2$

$$\pi/2 = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin n\pi/2$$

$$\frac{\pi}{2} = \frac{(-2)(-1)}{1}(1) + \frac{(-2)(-1)^2}{2}\sin\frac{2\pi}{2} + \frac{(-2)(-1)^3}{3}\sin\frac{3\pi}{2} + \frac{(-2)(-1)^4}{4}\sin 4\pi/2 + \frac{(-2)(-1)^5}{5}\sin 5\pi/2 + \cdots$$

$$= 2 + (-1)(0) + \frac{(-2)(-1)(-1)}{3} + \frac{(-2)}{4} \times 0 + \frac{(-2)(-1)}{5} \times 1 + \cdots$$

$$= 2 + 0 - \frac{2}{3} + 0 + \frac{2}{5}$$

$$\pi/2 = 2\left(1 - \frac{1}{3} + \frac{1}{5} + \cdots\right)$$

$$\frac{\pi}{4} = 1 - 1/3 + 1/5 - \frac{1}{7} + \cdots$$

* $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $0 < x < 2\pi$, S.T

$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ hence deduce

that ① $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

② $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

③ $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

Soln: The fourier series of period 2π is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ —①

consider $f(x) = \left(\frac{\pi - x}{2}\right)^2$

$f(2\pi - x) = \left(\frac{\pi - (2\pi - x)}{2}\right)^2$

$= \left(\frac{-\pi + x}{2}\right)^2$

$= \left(\frac{-(\pi - x)}{2}\right)^2$

$= \left(\frac{\pi - x}{2}\right)^2$

$f(2\pi - x) = f(x)$

$f(x)$ is even function

$b_n = 0$

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi} \int_0^{\pi} \frac{(\pi - x)^2}{4}\,dx$

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$$a_0 = \frac{2}{\pi}\int_{\pi}^{0} \frac{t^2}{4} \times -dt$$

put $\pi - x = t$

$0 \to x$

$-dx = dt$

$dx = -dt$

$x=0 \quad t=\pi$

$x=\pi \quad t=0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \frac{t^2}{4}\, dt$$

$$= \frac{2}{\pi} \times \left.\frac{t^3}{12}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \times \frac{\pi^3}{12}$$

$$\underline{a_0 = \pi^2/6}$$

(or) $a_0 = \frac{2}{\pi}\left[\frac{(\pi-x)^3}{4\times -3}\right]_0^{\pi}$

$$= \frac{2}{\pi}\left[\frac{1}{-12}\left(0-(\pi)^3\right)\right.$$

$$= \frac{2}{\pi} \times \frac{-\pi^3}{-12}$$

$$\underline{a_0 = \pi^2/6}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{(\pi-x)^2}{4}\cos nx\, dx$$

$$= \frac{2}{4\pi}\int_0^{\pi} (\pi-x)^2\cos nx\, dx$$

Apply Bernoulli's rule

$$a_n = \frac{1}{2\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2(-1)(-1)\left(\frac{-1}{n^2}\times\frac{\sin nx}{n}\right)\right]$$

$$= \frac{1}{2\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)\frac{\cos nx}{n^2} - \frac{\sin nx}{n^3}\right]_0^{\pi}$$

$\sin n\pi = 0 = \sin 0$

$$= \frac{1}{2\pi}\left[-2(\pi-x)\frac{\cos nx}{n^2}\right]_0^{\pi}$$

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$$= \frac{1}{2\pi}\left[-2\cancel{(\pi-\pi)}^{0} \frac{\cos n\pi}{n^2} + 2(\pi - 0)\frac{\cos(0)}{n^2} \right]$$

$$= \frac{1}{\cancel{2\pi}}\left[\frac{\cancel{2\pi}\cos 0}{n^2} \right]$$

$$a_n = \frac{1}{n^2}$$

put a_0, a_n, b_n in (1)

$$f(x) = \frac{\pi^2/6}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \quad \text{—} (*)$$

To deduce given series put $x = 0$

$$\pi^2/4 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} 1/n^2$$

$$\pi^2/4 - \pi^2/12 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad \text{—} (i)$$

put $x = \pi$ in (*)

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$$

$$\frac{-\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

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$$\frac{-\pi^2}{12} = \frac{-1}{1^2} + \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \cdots$$

$$\frac{-\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \quad \text{(ii)}$$

add (i) & (ii)

$$\frac{3\pi^2}{12} = 2\left(\frac{1}{1^2}\right) + 2\left(\frac{1}{3^2}\right) + 2\left(\frac{1}{5^2}\right) + \cdots$$

$$\frac{3\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{3\pi^2}{24} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\pi^2/8 = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Fourier Series of arbitary period

A function $f(x)$ need not always be defined in the interval of length 2π only. when the length of the interval is other than 2π we shall denote it by $2l$. A general interval of length $2l$ be $(c, c+2l)$

Here it is important to note that the sine & cosine functions of the form $\sin\left(\frac{\pi x}{l}\right)$ and $\cos\left(\frac{\pi x}{l}\right)$ are periodic functions of period $2l$.

Let $f(x) = \sin \frac{\pi x}{l}$

$$f(x+2l) = \sin\left(\frac{\pi}{l}(x+2l)\right)$$

$$= \sin\left(\frac{\pi x}{l} + 2\pi\right)$$

$$= \sin\left(2\pi + \frac{\pi x}{l}\right)$$

$$f(x+2l) = f(x)$$

III^{ly} $g(x+2l) = g(x)$

hence fourier series of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

NOTE: If the given function is neither odd nor even function then fourier Co-efficients are

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \qquad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Case ①: If $f(x)$ is an even function

$$f(-x) = f(x)$$
$$f(2l - x) = f(x)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = 0$$

Case ② If $f(x)$ is an odd function

$$f(-x) = -f(x)$$

$$f(2l - x) = -f(x)$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

NOTE: In every problem equateng given interval to $2l$

Problems

① obtain the fourier series to represent $f(x) = x - x^2$ in $-1 < x < 1$

Solⁿ: Here period of $f(x) = 1 - (-1) = 2$

$$\cancel{2}l = \cancel{2}$$
$$l = 1$$

Fourier series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

put $l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad — ①$$

Now we shall check the nature of the function is even or odd

$f(x) = x - x^2$

$f(-x) = -x - (-x)^2$

$f(-x) = -x - x^2$

$f(-x)$ is not equal to $f(x)$. or $-f(x)$

i.e which is neither even nor odd function

So, we have to find fourier co-efficients

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx \qquad a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

w.K.T $\boxed{l=1}$

$$a_0 = \frac{1}{1}\int_{-1}^{1}(x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{(-1)}{3}\right)$$

$$= \frac{1}{6} - \left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{1}{6} - \frac{5}{6}$$

$$= \frac{-4}{6}$$

$$\boxed{a_0 = \frac{-2}{3}}$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_{-1}^{1}(x - x^2)\cos n\pi x\,dx$$

Apply Bernoulli's rule

$$= \left[(x-x^2)\frac{\sin n\pi x}{n\pi} - (1-2x)\times\frac{+1}{n\pi}\times\frac{\cos n\pi x}{n\pi} + (0-2)\times\frac{-1}{n^2\pi^2}\times\frac{\sin n\pi x}{n\pi}\right]_{-1}^{1}$$

$$= \left[\cancel{(x-x^2)\frac{\sin n\pi x}{n\pi}}^{\to 0} + (1-2x)\frac{\cos n\pi x}{n^2\pi^2} + \underbrace{\cancel{2\frac{\sin n\pi x}{n^3\pi^3}}}_{0}\right]_{-1}^{1}$$

$\sin n\pi = 0$

$$= \left[(1-2x)\frac{\cos n\pi x}{n^2\pi^2}\right]_{-1}^{1}$$

$$= \left[\left((-1)\frac{\cos n\pi}{n^2\pi^2}\right) - \left(\frac{3\cos n\pi(-1)}{n^2\pi^2}\right)\right]$$

$$= \left[\frac{-\cos n\pi}{n^2\pi^2} - \frac{3\cos n\pi}{n^2\pi^2}\right]$$

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$$= \frac{-4\cos n\pi}{n^2\pi^2}$$

$$= \frac{-4(-1)^n}{n^2\pi^2}$$

$$= \frac{4 \times -1 \times (-1)^n}{n^2\pi^2}$$

$$= \frac{4(-1)^1(-1)^n}{n^2\pi^2}$$

$$a_n = \frac{4(-1)^{n+1}}{n^2\pi^2} \quad //$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{1}\int_{-1}^{1} (x-x^2)\sin n\pi x\,dx$$

Apply Bernoulli's rule

$$= \left[(x-x^2) \times \frac{-\cos n\pi x}{n\pi} - (1-2x) \times \frac{-1}{n\pi} \times \frac{\sin n\pi x}{n\pi} + (0-2) \times \frac{-1}{n^2\pi^2} \times \frac{-\cos n\pi x}{n\pi} \right]_{-1}^{1}$$

$$= \left[-(x-x^2)\frac{\cos n\pi x}{n\pi} + \underbrace{\frac{(1-2x)\sin n\pi x}{n^2\pi^2}}_{0} - \frac{2\cos n\pi x}{n^3\pi^3} \right]_{-1}^{1}$$

$$\because \sin n\pi = 0$$

$$= \left[-(x-x^2)\frac{\cos n\pi x}{n\pi} - \frac{2\cos n\pi x}{n^3\pi^3} \right]_{-1}^{1}$$

$$= \left[\left\{ \underbrace{-(1-1)\frac{\cos n\pi}{n\pi}}_{0} - \frac{2\cos n\pi}{n^3\pi^3} \right\}_{\text{upper limit}} - \left\{ -(-1-1)\frac{\cos n\pi(-1)}{n\pi} - \frac{2\cos n\pi(-1)}{n^3\pi^3} \right\}_{\text{lower limit}} \right]$$

$$= \left[\left\{ \frac{-2(-1)^n}{n^3\pi^3} \right\} - \left\{ -x - 2\frac{\cos n\pi}{n\pi} - 2\frac{\cos n\pi}{n^3\pi^3} \right\} \right]$$

$$= \cancel{\frac{-2(-1)^n}{n^3\pi^3}} - \frac{2(-1)^n}{n\pi} + \cancel{\frac{2(-1)^n}{n^3\pi^3}}$$

$$= \frac{-2(-1)^n}{n\pi}$$

$$= \frac{2 \times -1 \times (-1)^n}{n\pi}$$

$$= \frac{2(-1)^1(-1)^n}{n\pi}$$

$$b_n = \frac{2(-1)^{n+1}}{n\pi}$$

put a_0, a_n, b_n in equation (1)

hence required fourier series is given by

$$f(x) = \frac{-2/3}{\cancel{2}} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2} \cos n\pi x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin n\pi x$$

$$f(x) = \frac{-1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi^2} \cos n\pi x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$$

② Obtain the fourier series of $f(x) = |x|$ in $(-l, l)$ hence show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}$

Solⁿ: The period of $f(x) = l-(-l) = 2l$ and fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{--- ①}$$

we shall check $f(x) = |x|$ is even or odd function

$$f(-x) = |-x| = |x| = f(x)$$

$$\therefore f(-x) = f(x)$$

hence $f(x)$ is an even function

$\therefore b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_0 = \frac{2}{l}\int_0^l x\,dx$$

$$= \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{2} - 0\right]$$

$$= \frac{\cancel{2}}{\cancel{l}} \times \frac{l^{\cancel{2}}}{\cancel{2}}$$

$$\boxed{a_0 = l}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

Apply Bernoulli's rule

$$a_n = \frac{2}{l}\left[\frac{x \sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} - (1) \times \frac{1}{\frac{n\pi}{l}} \times \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right]_0^l$$

$$a_n = \frac{2}{l}\left[\cancel{\frac{x \sin \frac{n\pi x}{l}}{n\pi/l}}^{\to 0} + \frac{\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right]_0^l$$

$\sin n\pi = 0 = \sin 0$

$$a_n = \frac{2}{\cancel{l} \times \frac{n^2\pi^2}{l^{\cancel{2}}}} \left[\cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{2}{\frac{n^2\pi^2}{l}} \left[\cos \frac{n\pi l}{l} - \cos(0) \right]$$

$$= \frac{2l}{n^2\pi^2} \left[\cos n\pi - 1 \right]$$

$$= \frac{2l}{n^2\pi^2} \left[(-1)^n - 1 \right]$$

$$= \frac{-2l}{n^2\pi^2} \left[1 - (-1)^n \right]$$

Required fourier series is given by

$$f(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{-2l}{n^2\pi^2} (1-(-1)^n) \cos \frac{n\pi x}{l}$$

To deduce the series put $x = 0$

w.k.T $f(x) = x$

$f(0) = 0$

$$0 = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{-2l}{n^2\pi^2} \{1-(-1)^n\} \cos(0)$$

$$-\frac{l}{2} = -\frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \{1-(-1)^n\} (1)$$

$$(1-(-1)^n) = \begin{cases} 1-(+1) = 0 & \text{when } n \text{ is even} \\ 1-(-1) = 2 & \text{when } n \text{ is odd} \end{cases}$$

$$\cancel{+}\frac{l}{2} = \cancel{+}\frac{2l}{\pi^2} \sum_{n=1,3,5..}^{\infty} \frac{1}{n^2} \times 2$$

$$\frac{\pi^2}{4} = 2 \sum_{n=1,3,5\ldots}^{\infty} 1/n^2$$

÷ Bothsides by 2

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

3) Obtain the fourier series for the function

$$f(x) = \begin{cases} 1 + \frac{4x}{3} & \text{in } -\frac{3}{2} < x \le 0 \\ 1 - \frac{4x}{3} & \text{in } 0 \le x < 3/2 \end{cases}$$

hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Solⁿ: $f(x)$ is defined in the interval $(-3/2, 3/2)$

$\therefore$ period of $f(x) = 3/2 - (-3/2) = 3$

$2l = 3$ or $l = 3/2$

we shall check $f(x)$ for even or odd nature

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$f(x) = 1 + \frac{4x}{3}$

$f(-x) = 1 - \frac{4x}{3}$

$f(-x) = f(x)$

$\therefore f(x)$ is even function
consequently $b_n = 0$

The fourier series having period 3 is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

put $l = \frac{3}{2}$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3/2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3/2} \quad \text{①}$$

Now,

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$l = 3/2$

$$= \frac{2}{3/2}\int_0^{3/2} f(x)\,dx$$

$$= 4/3 \int_0^{3/2} \left(1 - \frac{4x}{3}\right) dx$$

$$= 4/3 \left[x - \frac{4}{3} \times \frac{x^2}{2} \right]_0^{3/2}$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{\cancel{4}^{2}}{\cancel{6}_{3}} \times \left(\frac{3}{2}\right)^2 - 0\right]$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{2}{3} \times \frac{9}{\cancel{4}_2}\right]$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{9}{6}\right]$$

$$= \frac{4}{3}\left[\frac{9-9}{6}\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3/2}\int_0^{3/2}\left(1 - \frac{4x}{3}\right)\cos\frac{n\pi x}{3/2}\,dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1 - \frac{4x}{3}\right)\cos\left(\frac{2n\pi x}{3}\right)dx$$

Apply Bernoulli's rule

$$= \frac{4}{3}\left[\left(1 - \frac{4x}{3}\right)\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} - \left(0 - \frac{4}{3}\right)\times\frac{1}{\frac{2n\pi}{3}}\times\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}}\right]_0^{3/2}$$

$$= \frac{4}{3}\left[\left(1 - 4x/3\right)\frac{\sin\left(\frac{2n\pi x}{3}\right)}{2n\pi/3} \underset{\text{minus}}{+} \frac{4}{3}\times\frac{1}{\left(\frac{2n\pi}{3}\right)^2}\cos\left(\frac{2n\pi x}{3}\right)\right]_0^{\frac{3}{2}}$$

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$$= \frac{4}{3}\left[\left\{\left(1-\frac{4x}{3}\right)\frac{\cancel{\sin}\left(\frac{2n\pi\cdot 3/2}{3}\right)}{\frac{2n\pi}{3}} - \frac{4}{3}\times\frac{9}{4\pi^2 n^2}\cos\left(\frac{2n\pi\cdot 3/2}{3}\right)\right\}\right.$$

$$\left. -\left\{\left(1-\frac{4(0)}{3}\right)\frac{\cancel{\sin}\left(\frac{0}{3}\right)}{2n\pi/3} - \frac{4}{3}\times\frac{9}{4\pi^2 n^2}\cos\left(\frac{0}{3}\right)\right\}\right]$$

0 0

$$= \frac{4}{3}\left[-\frac{\cancel{4}}{\cancel{3}}\times\frac{\cancel{9}^{3}}{\cancel{4}\pi^2 n^2}\cos n\pi + \frac{\cancel{4}}{\cancel{3}}\times\frac{\cancel{9}^{3}}{\cancel{4}\pi^2 n^2}\times\cos(0)\right]$$

$$= \frac{4}{3}\left[\frac{-3}{\pi^2 n^2}\cos n\pi + \frac{3}{\pi^2 n^2}\times 1\right]$$

$$= \frac{12}{3\pi^2 n^2}\left[1-\cos n\pi\right]$$

$$= \frac{\cancel{12}^{4}}{\cancel{3}\pi^2 n^2}\left[1-(-1)^n\right]$$

$$\boxed{a_n = \frac{4}{\pi^2 n^2}\left(1-(-1)^n\right)}$$

Direct go here

This required fourier series is given by

i.e put a_0, a_n, b_n in ①

$$f(x) = 0 + \sum_{n=1,3\ldots}^{\infty} \frac{8}{\pi^2 n^2} \cos\left(\frac{n\pi x}{3/2}\right) + 0$$

$$f(x) = 1 - \frac{4x}{3}$$

$$1 - \frac{4x}{3} = \frac{8}{\pi^2} \sum_{n=1,3}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{3}\right)$$

put $x = 0$

$$1 = \frac{8}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \cos(0)$$

$$1 = \frac{8}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \times 1 \Rightarrow \frac{\pi^2}{8} = \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$$a_n = \frac{4}{\pi^2 n^2}\left(1 - (-1)^n\right)$$

put a_0, a_n, b_n in ①

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left(1 - (-1)^n\right)$$

$$1 - (-1)^n = \begin{cases} 1 - (+1) = 0 & \text{when } n \text{ is even} \\ 1 - (-1) = 2 & \text{when } n \text{ is odd} \end{cases}$$

$f(x) = 1 - 4x/3$

$$1 - \frac{4x}{3} = \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{\pi^2 n^2} \times 2$$

put $x = 0$

$$1 = \frac{8}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

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(4) If $f(x) = \begin{cases} 2-x & \text{in } 0 \le x \le 4 \\ x-6 & \text{in } 4 \le x \le 8 \end{cases}$

Express $f(x)$ by a fourier series and hence deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Solⁿ: $f(x)$ is defined in the interval $(0, 8)$

$\therefore$ period of $f(x) = 8 - 0 = 8$

$$2l = 8$$

$$l = 4$$

Fourier series having period 8 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

put $l = 4$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$$

we shall check $f(x)$ for even or odd nature

$$f(x) = 2 - x$$

$$\begin{aligned} f(2l-x) &= 2 - (2l - x) \\ &= 2 - 2l + x \\ &= 2 - 2\times 4 + x \\ &= 2 - 8 + x \\ &= -6 + x \\ &= x - 6 \end{aligned}$$

$$f(2l-x) = f(x)$$

(or) $f(x) = x - 6$

$$\begin{aligned} f(2l-x) &= 2l - x - 6 \\ &= 2\times 4 - x - 6 \\ &= 8 - 6 - x \\ &= 2 - x \end{aligned}$$

$$f(-x) = f(x)$$

②

2.

$f(x)$ is even function

clearly $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{4}\int_0^4 (2-x)\,dx$$

$$= \frac{1}{2}\left[2x - \frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2}\left[2\times 4 - \frac{4^2}{2} - 0\right]$$

$$= \frac{1}{2}\left[8 - \cancel{\frac{16}{2}}^{\,8}\right]$$

$$= \frac{1}{2}(8-8)$$

$$\underline{a_0 = 0}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_n = \frac{2}{4}\int_0^4 (2-x)\cos\left(\frac{n\pi x}{4}\right)dx \quad \text{, Apply Bernoulli's rule}$$

$$= \frac{1}{2}\left[\cancel{\frac{(2-x)\sin\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}} \to 0 - (0-1)\frac{1}{\frac{n\pi}{4}} \times \frac{-\cos\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}\right]_0^4$$

$$= \frac{1}{2}\left[-\frac{1}{\frac{n^2\pi^2}{16}} \times \cos\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= -\frac{1}{\cancel{2}} \times \frac{\cancel{16}^{\,8}}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= \frac{-8}{n^2\pi^2}\left[\cos\left(\frac{n\pi \cancel{4}}{\cancel{4}}\right) - \cos(0)\right]$$

$$= \frac{-8}{\pi^2 n^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-8}{\pi^2 n^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{8}{\pi^2 n^2}\left[1 - (-1)^n\right]$$

put a_0, a_n, b_n in (1)

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{8}{\pi^2 n^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{4}\right) + 0$$

$$2 - x = \sum_{n=1}^{\infty} \frac{8\left(1-(-1)^n\right)}{\pi^2 n^2}\cos\left(\frac{n\pi x}{4}\right)$$

put $x=0$

$$1-(-1)^n = \begin{cases} 0 & \text{when } n \text{ is even} \\ 2 & \text{when } n \text{ is odd} \end{cases}$$

$$2 = \frac{8}{\pi^2}\sum_{n=1,3..}^{\infty} \frac{2}{n^2}\cdot\cos(0)$$

$$\cancel{2} = \frac{8}{\pi^2}\times\cancel{2}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^2}\times 1$$

$$\frac{\pi^2}{8} = \sum_{n=1,3,5...}^{\infty} 1/n^2$$

$$\pi^2/8 = 1/1^2 + 1/3^2 + 1/5^2 + \cdots$$

equivalently $\pi^2/8 = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

(5) obtain the fourier series for the function $f(x) = \begin{cases} \pi x & \text{in } 0 \le x \le 1 \\ \pi(2-x) & \text{in } 1 \le x \le 2 \end{cases}$

& S.T the fourier series expansion of the function $f(x)$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots\right)$$

Soln:

$f(x)$ is defined in the interval $(0,2)$

Period of $f(x) = 2-0 = 2$

$$2l = 2$$

$$l = 1$$

we shall check the nature of the function

$f(x) = \pi x$

$f(2l-x) = \pi(2l-x)$

$l=1$

$f(2l-x) = \pi(2-x)$

$f(2l-x) = f(x)$

(or) $f(x) = \pi(2-x)$

$f(2l-x) = \pi\,(\cancel{2-2l+x})$

$= \pi(\cancel{2-2l+2x})$

$= \pi(\cancel{2-2x})$

$= \pi(2-(2l-x))$

$= \pi(2-2l+x)$

$= \pi(2-2+x)$

$= \pi x$

$\therefore f(x)$ is even function $f(2l-x) = f(x)$

fourier series having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$l=1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{--- (1)}$$

$f(x)$ is even function

clearly $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 \pi x\,dx$$

$$= 2\cdot\pi\left[\frac{x^2}{2}\right]_0^1$$

$$= 2\pi\left(\frac{1}{2}-0\right)$$

$$= \frac{\cancel{2}\pi}{\cancel{2}}$$

$a_0 = \pi$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$\boxed{l=1}$

$$= 2\int_0^1 f(x)\cos n\pi x\,dx$$

$$= 2\int_0^1 \pi x\cos n\pi x\,dx$$

$$= 2\pi\left[\frac{x\,\cancel{\sin n\pi x}}{n\pi} - (1)\times -\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= 2\pi\left[\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= 2\pi\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{\cos 0}{n^2\pi^2}\right]$$

$$= \frac{2\pi}{n^2\pi^2}(\cos n\pi - 1)$$

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$$= \frac{2}{n^2\pi^2}((-1)^n - 1)$$

$$a_n = \frac{-2}{\pi n^2}(1-(-1)^n)$$

put a_0, a_n, b_n in ①

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{\pi n^2}(1-(-1)^n)\cos n\pi x$$

$$(1-(-1)^n) = \begin{cases} 2 & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{\cos n\pi x}{n^2}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$

put $x = 0$ If they asked Series

$f(x) = \pi x$ $\therefore f(0) = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos(0)}{1^2} + \frac{\cos(0)}{3^2} + \frac{\cos(0)}{5^2} + \ldots\right]$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

⑥ Obtain the fourier Series for the function $f(x) = 2x - x^2$ in $0 \le x \le 2$

Do yourself

Soln: $l = 1$, $b_n = 0$ $\because$ even function

Replace x by $2l - x$

$$f(x) = 2x - x^2 = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2}\cos(n\pi x)$$

Half Range Fourier Series

In an interval of length $2l$, we have seen that in general a periodic function of x will have fourier expansion containing cosine terms & sine terms. Many times it becomes necessary to have the expansion containing only cosine terms or sine terms

To achieve this the function must be defined in the interval of the form $(0, l)$ & $(0, \pi)$ which is to be regarded as half the interval. We then extend the definition to other half in such a manner that the function becomes even or odd. This will result in cosine series only or sine series only

Case ①: In the interval $(0, l)$
cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{— ①}$$

Fourier co-efficient $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_n = 2/l \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx ,$$

Case ②: In the interval $(0, l)$
Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{— ②}$$

Fourier co-eff $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$

Case ③: In the interval $(0,\pi)$

Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—③}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

Case ④: In the interval $(0,\pi)$

Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{—④}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

① and ③ are called cosine half range fourier series, because it contains only cosine terms

② & ④ are called sine half range fourier series because it contains only sine terms.

Problems

NOTE: In half range fourier series to calculate period, equate to l

① Expand $f(x) = 2x - 1$ as a cosine half range fourier series in $0 < x < 1$

Sol^{n}: Comparing the given interval $(0, 1)$ with $(0, l)$

we have $l = 1$

(or) The given function having period

$f(x) = 1 - (-0) = 1$

$\therefore l = 1$

Cosine half range fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 (2x-1)\,dx$$

$$= 2\left[\frac{2 \cdot x^2}{2} - x\right]_0^1$$

$$= 2\left[x^2 - x\right]_0^1$$

$$= 2\left[(1-1) - 0\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{1}\int_0^1 (2x-1)\cos n\pi x\,dx \quad (\because l = 1)$$

$$= 2\left[(2x-1)\cancelto{0}{\frac{\sin n\pi x}{n\pi}} - (2) \times \frac{-\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= 2\left[\frac{2\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left(\cos n\pi(1) - \cos(0)\right)$$

$$= \frac{4}{\pi^2 n^2}\left(\cos n\pi - 1\right)$$

$$= \frac{4}{\pi^2 n^2}\left((-1)^n - 1\right)$$

$$a_n = \frac{-4}{\pi^2 n^2}\left(1-(-1)^n\right)$$

$\therefore$ cosine half range Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{-4}{\pi^2 n^2}\left(1-(-1)\right)^n \cos n\pi x$$

② S.T the sine half range series for the function $f(x) = lx - x^2$ in $0 < x < l$ is $\frac{8l^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left(\frac{2n+1}{l}\right)\pi x$

Soln: The sine half range Fourier series of $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

Apply Bernoulli's rule

upper limit $l(l)-l^2 = l^2-l^2 = 0$
lower limit $\rightarrow 0$

$$= \frac{2}{l}\left[\cancel{(lx-x^2)}\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} - \cancel{(1-2x)}\times\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} + (-2)\times\frac{-\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$\therefore \sin n\pi = 0 = \sin 0$

$$= -\frac{2}{l}\left[\frac{2\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \frac{-4}{l}\times\frac{l^3}{n^3\pi^3}\left[\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-4l^2}{n^3\pi^3}\left[\cos n\pi - \cos(0)\right]$$

$$= \frac{-4l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$= \frac{4l^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

The sine half range fourier series is given by

$$f(x) = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1-(-1)^n\right)\sin\frac{n\pi x}{l}$$

$$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{when } n \text{ is even} \\ 1-(-1) = 2 & \text{when } n \text{ is odd}\end{cases}$$

$$f(x) = \frac{4l^2}{\pi^3}\sum_{n=1,3,5\ldots}^{\infty}\frac{2}{n^3}\sin\frac{n\pi x}{l}$$

$$= \frac{8l^2}{\pi^3}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}$$

But 1, 3, 5 ... are odd numbers represented in general as $(2n+1)$ where $n = 0, 1, 2, 3 \ldots\ldots$ $\therefore$ we have

$$f(x) = \frac{8l^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin\left(\frac{2n+1}{l}\right)\pi x$$

③ obtain the sine half range fourier series of $f(x) = x^2$ in $0 < x < \pi$

Solⁿ: The sine half range fourier series of the function $f(x)$ in $(0,\pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx$$

Apply Bernoulli's rule.

$$= \frac{2}{\pi}\left[x^2\times\frac{-\cos nx}{n} - (2x)\times\frac{-\sin nx}{n^2} + (2)\times\frac{-\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x^2\cos nx}{n} + \cancel{\frac{2x\sin nx}{n^2}} + \frac{2\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right) - \left(0 + \frac{2\cos(0)}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - \frac{2}{n^3}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}(\cos n\pi - 1)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}((-1)^n - 1)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \times (-1)^1(-1)^n}{n} + \frac{-2}{n^3}(1-(-1)^n)\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} - \frac{2}{n^3}(1-(-1)^n)\right]$$

required half range fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} - \frac{2}{n^3}(1-(-1)^n)\right]\sin nx$$

(4) obtain the sine half range series of

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{in } 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \text{in } \frac{1}{2} < x < 1 \end{cases}$$

Solⁿ: $f(x)$ is defined in the interval $(0,1)$ comparing with half range $(0, l)$

$\therefore l = 1$

$\therefore$ sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$b_n = 2\int_0^1 f(x) \sin n\pi x\, dx \qquad (\because l = 1)$$

Because

$$b_n = 2\left\{\int_0^{1/2} f(x) \sin n\pi x\, dx + \int_{1/2}^{1} f(x) \sin n\pi x\, dx\right\}$$

$$= 2\left\{\int_0^{1/2}\left(\tfrac{1}{4}-x\right)\sin n\pi x\,dx + \int_{1/2}^{1}\left(x-\tfrac{3}{4}\right)\sin n\pi x\,dx\right\}$$

$$= 2\left\{\left[\left(\tfrac{1}{4}-x\right)\times\frac{-\cos n\pi x}{n\pi} - (-1)\times\frac{-\sin n\pi x}{n^2\pi^2}\right]_0^{1/2}\right.$$

$$\left. + \left[\left(x-\tfrac{3}{4}\right)\times\frac{-\cos n\pi x}{n\pi} - (1)\times -\times\frac{\sin n\pi x}{n^2\pi^2}\right]_{1/2}^{1}\right.$$

$$= 2\left\{\left[-\left(\tfrac{1}{4}-x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^{1/2} + \right.$$

$$\left.\left[\left(x-\tfrac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right]_{1/2}^{1}\right\}$$

$$= 2\left\{-\left(\tfrac{1}{4}-\tfrac{1}{2}\right)\frac{\cos n\pi\left(\tfrac{1}{2}\right)}{n\pi} - \frac{\sin n\pi\left(\tfrac{1}{2}\right)}{n^2\pi^2}\right.$$

$$- \left(-\left(\tfrac{1}{4}-0\right)\frac{\cos(0)}{n\pi} - \cancel{\frac{\sin(0)}{n^2\pi^2}}^{\to 0}\right)$$

$$+ \left(\left(1-\tfrac{3}{4}\right)\left(-\frac{\cos n\pi}{n\pi}\right) + \cancel{\frac{\sin(n\pi)}{n^2\pi^2}}^{\to 0}\right) -$$

$$\left.\left(\tfrac{1}{2}-\tfrac{3}{4}\right)\left(-\frac{\cos n\pi\left(\tfrac{1}{2}\right)}{n\pi}\right) + \frac{\sin n\pi\left(\tfrac{1}{2}\right)}{n^2\pi^2}\right\}$$

$$= 2\left\{\frac{-1}{n\pi}\left(-\tfrac{1}{4}\right)\cos\frac{n\pi}{2} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{1}{4}\times\frac{1}{n\pi}\right.$$

$$\left. -\frac{1}{4n\pi}(-1)^n + \left(-\tfrac{1}{4}\right)\frac{1}{n\pi}\times\cos\frac{n\pi}{2} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= 2\left\{\frac{1}{4n\pi}\left(\cancel{\cos\frac{n\pi}{2}} + 1 - (-1)^n - \cancel{\cos\frac{n\pi}{2}}\right)\right.$$

$$\left. - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= 2\left\{\frac{1}{4n\pi}\left(1-(-1)^n\right) - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

Sine half range Series is given by

$$f(x) = \sum_{n=1}^{\infty} 2\left\{ \frac{1}{4n\pi}(1-(-1)^n) - \frac{2}{\pi^2 n^2}\sin\frac{n\pi}{2} \right\} \sin\frac{n\pi x}{l}$$

(5) Find the cosine half range Series of $f(x) = x\sin x$ in $0 < x < \pi$. Deduce that $\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \dots = \frac{\pi-2}{4}$ or

$$1 + \frac{2}{1\cdot3} - \frac{2}{3\cdot5} + \frac{2}{5\cdot7} \dots = \frac{\pi}{2}$$

Soln: Cosine half range Series of $f(x)$ in the half range $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—— (1)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\Big[x \times -\cos x - (1)(-\sin x)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[(-\pi\cos\pi + \cancel{\sin\pi}^{\,0}) - 0\Big]$$

$$= \frac{2}{\cancel{\pi}} \times -\cancel{\pi}\cos\pi$$

$$= -2\times(-1) \qquad \because \cos\pi = \cos 180 = -1$$

$$a_0 = 2$$

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$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx \quad \text{--- } 2$$

$$\sin A\cos B = \tfrac{1}{2}\{\sin(A+B)+\sin(A-B)\}$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,\frac{1}{2}\{\sin(x+nx)+\sin(x-nx)\}\,dx$$

$$= \frac{2}{\pi}\times\frac{1}{2}\int_0^{\pi} x\{\sin(1+n)x+\sin(1-n)x\}\,dx$$

$$\sin((1-n)x) = \sin(-(n-1)x) = -\sin((n-1)x)$$

$$= \frac{1}{\pi}\int_0^{\pi} x\{\sin(n+1)x - \sin(n-1)x\}\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(n+1)x\,dx + \int_0^{\pi} -x\sin(n-1)x\,dx\right]$$

Apply Bernoulli's rule

$$= \frac{1}{\pi}\left\{\left[x\times\frac{-\cos(n+1)x}{(n+1)} - (1)\left(-\frac{1}{n+1}\right)\left(\frac{\sin(n+1)x}{(n+1)}\right)\right]_0^{\pi}\right.$$

$$\left. + \left[x\times\frac{-\cos(n-1)x}{(n-1)} - (1)\times\frac{-1}{(n-1)}\times\frac{\sin(n-1)x}{(n-1)}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x\cos(n+1)x}{n+1} + \underbrace{\cancel{\frac{\sin(n+1)x}{(n+1)^2}}}_{\to 0} + \frac{x\cos(n-1)x}{(n-1)} + \underbrace{\cancel{\frac{\sin(n-1)x}{(n-1)^2}}}_{0}\right]_0^{\pi}\right\}$$

$$\sin(n+1)\pi = 0 = \sin(n-1)\pi\,;\quad \sin 0 = 0$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{(n+1)} + \frac{\pi\cos(n-1)\pi}{(n-1)} - 0 - 0\right]$$

$$= \frac{1}{\cancel{\pi}}\times\cancel{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right]$$

$$= +\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \frac{(-1)^1(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

$$= \frac{(-1)^n \cdot (-1)^2}{n+1} + \frac{(-1)^n(-1)^{-1}}{n-1}$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{-1}{n-1}\right]$$

$a^{-1} = 1/a$

$(-a)^{-1} = -1/a$

$(-1)^{-1} = -1/1$

$$= (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right]$$

$$= \frac{(-1)^n(-2)}{n^2-1}$$

$$= \frac{2\times(-1)(-1)^n}{n^2-1}$$

$$a_n = \frac{2(-1)^{n+1}}{n^2-1} \quad \text{where } n \neq 1$$

we shall now find a_n when $n=1$, i.e to find a_1. Let us consider a_n as given by ② put $n=1$ we have

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$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x \, dx$$

(1) $\times^{ly}$ & $\div$ by 2

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x 2\times \sin x\cos x}{2} dx$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{x\sin 2x}{\cancel{2}} dx$$

$$= \frac{1}{\pi}\left[x\times \frac{-\cos 2x}{2} - (1)\left(\frac{-1}{2}\times \frac{\sin 2x}{2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos 2\pi}{2} + \frac{1}{2}\cancel{\frac{\sin 2\pi}{2}}^{0} - 0\right]$$

$\sin 2\pi = 0$

$$= \frac{1}{\cancel{\pi}}\times \frac{-\cancel{\pi}\cos 2\pi}{2}$$

$$a_1 = -\frac{1}{2}$$

Now eq (1) becomes

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx$$

$$= +\frac{\cancel{2}}{\cancel{2}} + \left(-\tfrac{1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx$$

$$f(x) = +1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

put $x = \pi/2$ $\quad f(x) = \pi/2 \sin \pi/2 = \pi/2 \times 1 = \pi/2$

$$\pi/2 = 1 + 0 + 2\sum_{n=2}^{\infty}\cos n\pi/2$$

$$\pi/2 - 1 = 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos n\pi/2$$

$$\frac{\pi-2}{2} = 2\left[\frac{\cos\pi\ (-1)}{3} + \frac{1}{8}\cancel{\cos\frac{3\pi}{2}}^{\,0} + \frac{(-1)}{15}\cos\frac{4\pi}{2} + \frac{(1)}{24}\cancel{\cos\frac{5\pi}{2}}^{\,0} + \dots\right]$$

$\cos\pi = -1 = \cos 3\pi = \cos 5\pi \dots$

$\cos(3\pi/2) = 0 = \cos 5\pi/2 \dots$

$$\frac{\pi-2}{2\times 2} = \frac{(-1)(-1)}{3} - \frac{1}{15}\times(1) + \dots$$

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \dots$$

(or) multiplying by 2 & transposing 1 on RHS we get

$$\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} \dots$$

Do yourself

(*) If $f(x) = \begin{cases} x & \text{in } 0 < x < \pi/2 \\ \pi - x & \text{in } \frac{\pi}{2} < x < \pi \end{cases}$

S.T (i) $f(x) = \frac{4}{\pi}\left\{\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 3x}{5^2} \dots\right\}$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \dots\right\}$

(*) Find a Cosine Series for $f(x) = (x-1)^2$, $0 \le x \le 1$

Ans: $l = 1$ $f(x) = 1/3 + 4/\pi^2 \sum_{n=1}^{\infty} 1/n^2 \cos n\pi x$

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(6) obtain half range cosine series for the function $f(x) = \sin\left(\frac{m\pi x}{l}\right)$ where m is a +ve integer over the interval $(0, l)$

Soln:
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{--- (1)}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\frac{-\cos\frac{m\pi x}{l}}{m\pi/l}\right]_0^l$$

$$= -\frac{2}{l} \times \frac{l}{m\pi}\left[\cos\left(\frac{m\pi x}{l}\right)\right]_0^l$$

$$= -\frac{2}{m\pi}\left[\cos m\pi - \cos 0\right].$$

$$= -\frac{2}{m\pi}\left[(-1)^m - 1\right]$$

$$a_0 = \frac{2}{m\pi}\left(1-(-1)^m\right)$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\cdot\cos\frac{n\pi x}{l}\,dx \quad \text{--- (2)}$$

$$= \frac{2}{l}\int_0^l \frac{1}{2}\left\{\sin\frac{(m+n)\pi x}{l} + \sin\frac{(m-n)\pi x}{l}\right\}dx$$

$$= 2/l\left[\frac{1}{2}\left\{\frac{-\cos\frac{(m+n)\pi x}{l}}{\frac{(m+n)\pi}{l}} - \frac{\cos\frac{(m-n)\pi x}{l}}{\frac{(m-n)\pi}{l}}\right\}\right]_0^l$$

$$= \frac{1}{l}\left[\frac{l}{(m+n)\pi} x - \cos\frac{(m+n)\pi x}{l} - \frac{l}{(m-n)\pi}\cos\frac{(m-n)\pi x}{l}\right]_0^l$$

$$= \frac{l}{l}\left[\frac{l}{(m+n)\pi} x - \cos\frac{(m+n)\pi l}{l} - \frac{l}{(m-n)\pi}\cos\frac{(m-n)\pi l}{l}\right.$$

$$\left. - \left\{\frac{-l}{(m+n)\pi}\cos(0) - \frac{l}{(m-n)\pi}\cos(0)\right\}\right]$$

$$= \frac{\cancel{l}}{\cancel{l}}\left[-\frac{\cos(m+n)\pi}{(m+n)\pi} - \frac{\cos(m-n)\pi}{(m-n)\pi}\right.$$

$$\left. + \frac{1}{(m+n)\pi} + \frac{1}{(m-n)\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{(m+n)} x - \cos(m+n)\pi - \frac{1}{(m-n)}\cos(m-n)\pi\right.$$

$$\left. + \frac{1}{m+n} + \frac{1}{m-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{m+n}\left\{\cos m\pi \cos n\pi - \sin m\pi \sin n\pi\right\}\right.$$

$$- \frac{1}{m-n}\left\{\cos m\pi \cos n\pi + \sin m\pi \sin n\pi\right\}$$

$$\left. + \frac{1}{m+n} + \frac{1}{m-n}\right]$$

$\sin m\pi = 0 = \sin n\pi$

$$= \frac{1}{\pi}\left[\frac{-1}{m+n} x \cos m\pi \cos n\pi - \frac{1}{m-n}\cos m\pi \cos n\pi\right.$$

$$\left. + \frac{1}{m+n} + \frac{1}{m-n}\right]$$

$$= \frac{1}{\pi}\left[-\cos m\pi \cos n\pi \left(\frac{1}{m+n} + \frac{1}{m-n}\right) + \frac{1}{m+n} + \frac{1}{m-n}\right.$$

$$\int_0^l \ldots = \frac{1}{\pi}\left[-(-1)^m(-1)^n\left(\frac{m-\cancel{n}+m+\cancel{n}}{m^2-n^2}\right)+\frac{m-\cancel{n}+m+\cancel{n}}{m^2-n^2}\right]$$

$$= \frac{1}{\pi}\left[(-1)^1(-1)^m(-1)^n\left(\frac{2m}{m^2-n^2}\right)+\frac{2m}{m^2-n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2m}{m^2-n^2}\left((-1)^1(-1)^m(-1)^n+1\right)\right]$$

$$a_n = \frac{2m}{\pi(m^2-n^2)}\left[(-1)^{1+m+n}+1\right] \quad \text{where } m \neq n$$

If $m=n$

eqn ② becomes

$$a_n = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \frac{2\sin\frac{n\pi x}{l}\cos\frac{n\pi x}{l}}{2}\,dx$$

(x^{ly} & $\div$ by 2)

$\sin 2x = 2\sin x\cos x$

$$= \frac{1}{l}\int_0^l \sin 2\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\cancel{l}}\left[\frac{-\cos\frac{2n\pi x}{l}}{2n\pi/\cancel{l}}\right]_0^l$$

$$= \frac{-1}{2n\pi}\left[\cos\frac{2n\pi l}{l}-\cos(0)\right]$$

$$= \frac{-1}{2n\pi}\left[\cos 2n\pi - 1\right]$$

$$= \frac{-1}{2n\pi}(1-1) \qquad \because \cos 2n\pi = 1$$

$\boxed{a_n = 0}$ when $m=n$

eqn (ii) becomes required cosine half range fourier series when $m \neq n$ is

$$f(x) = \frac{1}{m\pi}\left\{1-(-1)^m\right\} + \sum_{n=1}^{\infty} \frac{2m}{\pi(m^2-n^2)}\left\{1+(-1)^{m+n-1}\right\}\cos\frac{n\pi x}{l}$$

Practical Harmonic Analysis

Harmonic analysis is the process of finding the constant term and first few cosine and sine terms numerically.

The fourier series of period 2π of a function $y = f(x)$ will be of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

The fourier series of practical analysis is

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots + \dots$$

$a_0/2$ is constant term

$(a_1 \cos x + b_1 \sin x)$ & $(a_2 \cos 2x + b_2 \sin 2x)$ are first & second harmonics respectively

NOTE: Suppose we have a set of N values of $y = f(x)$ having period 2π at equidistant points of x in the interval $0 \le x < 2\pi$ or $0 < x \le 2\pi$

Pr

(1) Det
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$x°$
y

Solⁿ:
i.e
This

$x°$	y
0	2
45	3/2
90	1
135	1/2
180	0
225	1/2
270	1
315	3/2
Total	8.0

If the value of y at $x=0$ & $x=2\pi$ are given out of this interval we omit one of them.

$\left\{ \cos\frac{n\pi x}{l} \right\}$

Fourier co-eff a_0, a_n, b_n are

$$a_0 = \frac{2}{N} \sum y$$

$$a_n = \frac{2}{N} \sum_{n=1}^{\infty} y \cos nx$$

$$b_n = \frac{2}{N} \sum_{n=1}^{\infty} y \sin nx$$

Problems

① Determine the constant term and the first cosine & sine terms of fourier series expansion of y from the following data

x°	0	45	90	135	180	225	270	315
y	2	3/2	1	1/2	0	1/2	1	3/2

Soln: Here the interval of 0° to 360° i.e $0 \le x < 2\pi$. we have to find a_0, a_1, b_1 This requiry summation of y, $y\cos x$, $y\sin x$

x°	y	$\cos x$	$y\cos x$	$\sin x$	$y\sin x$
0	2	1	2	0	0
45	3/2	0.7071	1.06065	0.7071	1.06065
90	1	0	0	1	1
135	1/2	-0.7071	-0.35355	0.7071	0.35355
180	0	-1	0	0	0
225	1/2	-0.7071	-0.35355	-0.7071	-0.35355
270	1	0	0	-1	-1
315	3/2	0.7071	1.06065	-0.7071	-1.06065
Total	8.0		3.4142		0

$N = 8$

$a_0 = \frac{2}{N} \Sigma y \quad a_1 = \frac{2}{N} \Sigma y \cos x$

$b_1 = \frac{2}{N} \Sigma y \sin x$

from the table

$\Sigma y = 8 \quad \Sigma y \cos x = 3.4142 \quad \Sigma y \sin x = 0$

$a_0 = \frac{2}{\cancel{8}} (\cancel{8}.0) = 2$

$a_0 = 2$

$a_1 = \frac{2}{N} \Sigma y \cos x = \frac{2}{8} \times 3.4142 = 0.85355$

$a_1 = 0.85355$ //

$b_1 = \frac{2}{N} \Sigma y \sin x$

$= \frac{2}{8} (0)$

$b_1 = 0$

fourier series of y up to first harmonic is given by

$y = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$

$y = \frac{2}{2} + 0.85355 \cos x + 0$

$y = 1 + 0.85355 \cos x$

(2) The turning moment T on the crank shaft of a stream engine for the crank angle θ is given as follows

θ°	0	30	60	90	120	150	180	210
T	0	2.7	5.2	7	8.1	8.3	7.9	6.8

240	270	300	330
5.5	4.1	2.6	1.2

Expand T as a fourier series upto first harmonics

Sol[n]: Here the interval of θ is $0 \le \theta < 2\pi$ period T is 2π. we are required to find a_0, a_1, b_1. The corresponding formula are

$a_0 = \frac{2}{N} \Sigma T$ $a_1 = \frac{2}{N} \Sigma T\cos\theta$ $b_1 = 2/N \Sigma T\sin\theta$

$N = 12$ $\frac{2}{N} = \frac{1}{6}$

P.T.O

θ°	T	$\cos\theta$	$T\cos\theta$	$\sin\theta$	$T\sin\theta$
0	0	1	0	0	0
30	2.7	0.866	2.3382	0.5	1.35
60	5.2	0.5	2.6	0.866	4.5032
90	7.0	0	0	1	7.0
120	8.1	-0.5	-4.05	0.866	7.0146
150	8.3	-0.866	-7.1878	0.5	4.15
180	7.9	-1	-7.9	0	0
210	6.8	-0.866	-5.8888	-0.5	-3.4
240	5.5	-0.5	-2.75	-0.866	-4.763
270	4.1	0	0	-1	-4.1
300	2.6	0.5	1.3	-0.866	-2.2516
330	1.2	0.866	1.0392	-0.5	-0.6
	59.4		-20.4992		8.9032

$a_0 = \frac{2}{N}\sum T$, $a_1 = \frac{2}{N}\sum T\cos\theta$

$= \frac{1}{6}\times 59.4$ $= \frac{1}{6}\times(-20.4922)$

$a_0 = 9.9$ $= -3.4165$

$b_1 = \frac{2}{N}\sum T\sin\theta$

$= \frac{1}{6}(8.9032)$

$b_1 = 1.4839$

Fourier Series upto first harmonic is

$T = f(\theta) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$

$T = 4.95 - 3.4165\cos\theta + 1.4839\sin\theta$

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(3) Given the following table

x^0	0	60°	120°	180°	240°	300°
y	7.9	7.2	3.6	0.5	0.9	6.8

obtain fourier series neglecting term higher than first harmonics

i.e find only first harmonics

Soln: Here the interval of x is $0°$ to $360°$
i.e $0 \le x < 2\pi$
we are required to find a_0, a_1, b_1 only

x^0	y	$\cos x$	$y\cos x$	$\sin x$	$y\sin x$
0	7.9	1	7.9	0	0
60	7.2	0.5	3.6	0.866	6.2352
120	3.6	−0.5	−1.8	0.866	3.1176
180	0.5	−1	−0.5	0	0
240	0.9	−0.5	−0.45	−0.866	−0.7794
300	6.8	0.5	3.4	−0.866	−5.8888
Total	26.9		12.15		2.6846

$N = 6 \quad \frac{2}{N} = \frac{1}{3}$

$a_0 = \frac{2}{N}\sum y \quad a_0 = \frac{1}{3}(26.9) = 8.9667$

$a_1 = \frac{2}{N}\sum y\cos x = \frac{1}{3}(12.15) = 4.05$

$b_1 = \frac{2}{N}\sum y\sin x = \frac{1}{3}(2.6846) = 0.8949$

Fourier Series up to first harmonic is

$y = \frac{a_0}{2} + a_1\cos x + b_1\sin x$

$y = 4.48335 + (4.05\cos x + 0.8949\sin x)$

④ Express y as a Fourier series upto third harmonics given

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Soln: Here the interval of x is $0 \le x \le 2\pi$ and values of y at $x=0$ and $x=2\pi$ must be same by periodic property $f(x+2\pi) = f(x)$

value of y at $x=0$ & 2π are both given & we must omit one of them

let us omit last value (2π) By definition from the given problem interval is $0 \le x \le 2\pi$, we must satisfy $0 \le x < 2\pi$ ②

The values of x are

0, 60, 120, 180, 240, 300, $N=6$

$0 \le x \le 2\pi$

x	y	$y\cos x$	$y\cos 2x$	$y\cos 3x$	$y\sin x$	$y\sin 2x$	$y\sin 3x$
0	1.98	1.98	1.98	1.98	0	0	0
60	1.3	0.65	-0.65	-1.3	1.1258	1.1258	0
120	1.05	-0.525	-0.525	1.05	0.9093	-0.9093	0
180	1.3	-1.3	1.3	-1.3	0	0	0
240	-0.88	0.44	0.44	-0.88	0.76208	-0.76208	0
300	-0.25	-0.125	0.125	0.25	0.2165	0.2165	0
Total	4.5	1.12	2.67	-0.2	3.01368	-0.32908	0

$a_0 = \frac{2}{N} \sum y = \frac{1}{3}(4.5) = 1.5$

$a_1 = \frac{2}{N} \sum y\cos x = \frac{1}{3}(1.12) = 0.3733$

$a_2 = \frac{2}{N} \sum y\cos 2x = \frac{1}{3} \times 2.67 = 0.89$

$a_3 = \frac{2}{N} \sum y\cos 3x = \frac{1}{3}(-0.2) = -0.0667$

$b_1 = \frac{2}{N} \sum y\sin x = \frac{1}{3}(3.01368) = 1.00456$

$b_2 = \frac{2}{N} \sum y\sin 2x = \frac{1}{3}(-0.32908) = -0.1097$

$b_3 = \frac{2}{N} \sum y\sin 3x = \frac{1}{3}(0) = 0$

Fourier series upto third harmonic is

$$y = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + (a_3\cos 3x + b_3\sin 3x)$$

$$y = 0.75 + (0.3733\cos x + 1.00456\sin x) + (0.89\cos 2x - 0.1097\sin 2x) + (-0.0667\cos 3x)$$

Do yourself

⑤* Compute the first two harmonics of Fourier series of $f(x)$ given the following table

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)=y$	1	1.4	1.9	1.7	1.5	1.2	1

Sol^n: values of $f(x) = y$ in the interval $0 \le x \le 2\pi$ and hence the last value $f(2\pi) = 1$ which is same as $f(0)$ omit last value

x°	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1	1	1	0	0	1	1	0	0
60	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	+1.2124
120	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180	1.7	-1	1	0	0	-1.7	1.7	0	0
240	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	-1.299	1.299
300	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
totals						-1.1	-0.3	0.5196	-0.1732

$a_1 = \frac{2}{N}\sum y\cos x = -0.366$

$a_2 = \frac{2}{N}\sum y\cos 2x = -0.1$

$b_1 = \frac{2}{N}\sum y\sin x = 0.1732$

$b_2 = \frac{2}{N}\sum y\sin 2x = -0.05773$

First two harmonics are
$a_1\cos x + b_1\sin x$ & $a_2\cos 2x + b_2\sin 2x$

are $(-0.367\cos x + 0.1732\sin x)$ &

$(-0.1\cos 2x - 0.0577\sin 2x)$

Just they asking you to find first two harmonics

If they asked fourier series upto second harmonics then find a_0

Do yourself

(6) Find fourier series to represent $y(x)$ upto second harmonic from the following data

x°	30	60	90	120	150	180	210	240	270	300	330	360
y	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Sol^n: The period of $y(x)$ is 2π

interval is $0 < x \le 2\pi$

$y(x) = a_0/2 + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$

$a_0 = 4.2017$ $a_1 = -0.28$ $a_2 = -0.18$

$b_1 = 1.61$ $b_2 = -0.5$

y or $f(x) = ?$

Findout

NOTE: ① If the period is not 2π, we equate it with $2l$ to obtain the value of l.

② The summation of y; $y\cos\theta, y\cos 2\theta \ldots\ldots; y\sin\theta, y\sin 2\theta \ldots$ where $\theta = \frac{\pi x}{l}$ will be required to compute the desired harmonics.

⑦ Obtain the constant term and co-efficients of first cosine & sine terms in the fourier series expansion of y from the table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Sol^n; The values at 0, 1, 2, 3, 4, 5 are given

$N=6$

interval of x should be $0 \le x < 6$

length of interval is $6-0=6$

equate to $2l$

$2l = 6$

$l = 3$

Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

put $l=3$, series containing first harmonic is

$$y = f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}$$

writing $\frac{\pi x}{3} = \theta$, $y = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$

$N = 6 \quad \frac{2}{N} = \frac{1}{3}$

x	y	$\theta = \frac{\pi x}{3}$	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	9	0	1	9	0	0
1	18	60°	0.5	9	0.866	15.588
2	24	120°	-0.5	-12	0.866	20.784
3	28	180°	-1	-28	0	0
4	26	240°	-0.5	-13	-0.866	-22.516
5	20	300°	0.5	10	-0.866	-17.32
Total	125			-25		-3.464

$a_0 = \frac{2}{N} \sum y \Rightarrow 41.67$

$\therefore$ constant term $a_0/2 = 20.835$

$a_1 = \frac{2}{N} \sum y\cos\theta \approx -8.333$

$b_1 = \frac{2}{N} \sum y\sin\theta \approx -1.155$

co-eff of 1st cosine & sine terms are -8.333 & -1.155 respectively

If they asked fourier series upto first harmonic

$f(x) = a_0/2 + a_1 \cos\theta + b_1 \sin\theta$

$= 20.835 - 8.333\cos\theta - 1.155\sin\theta$

$= 20.835 - 8.333\cos(\pi x/l) - 1.155\sin(\pi x/l)$

w.k.T $l = 3$

$f(x) = 20.835 - 8.333\cos\left(\frac{\pi x}{3}\right) - 1.155\sin\left(\frac{\pi x}{3}\right)$

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Do yourself

(8)* Express y as a Fourier Series up to 3rd harmonics given the following data

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Solⁿ: The interval of x is $0 \le x < 6$

$2l = 6 \Rightarrow l = 3, \quad N = 6 \quad \frac{1}{N} = \frac{1}{3}$

Fourier Series upto 3rd harmonics

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right) + \left(a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l}\right)$$

where $l = 3$

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right) + \left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

put $\frac{\pi x}{3} = \theta$

$$y = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta)$$

Then construct table.

Find a_0, a_1, a_2, a_3

b_1, b_2, b_3

th Home work problem

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(6) The period of $y(x)$ is 2π i.e 360°

Fourier Series upto Second harmonic is

$$y(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

$$a_0 = \frac{2}{N}\sum y, \quad a_1 = \frac{2}{N}\sum y\cos x \quad a_2 = \frac{2}{N}\sum y\cos 2x$$

$$b_1 = \frac{2}{N}\sum y\sin x \quad b_2 = \frac{2}{N}\sum y\sin 2x$$

x°	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
30°	2.34	0.87	0.5	0.5	0.87	2.0358	1.17	1.17	2.0358
60°	3.01	0.5	-0.5	0.87	0.87	1.505	-1.505	2.6187	2.6187
90	3.68	0	-1	1	0	0	-3.68	3.68	0
120	4.15	-0.5	-0.5	0.87	-0.87	-2.075	-2.075	3.6105	-3.6105
150	3.69	-0.87	0.5	0.5	-0.87	-3.2103	1.845	1.845	-3.2103
180	2.20	-1	1	0	0	-2.2	2.2	0	0
210	0.83	-0.87	0.5	-0.5	0.87	-0.7221	0.415	-0.415	0.7221
240	0.51	-0.5	-0.5	-0.87	0.87	-0.255	-0.255	-0.4437	0.4437
270	0.88	0	-1	-1	0	0	-0.88	-0.88	0
300	1.09	0.5	-0.5	-0.87	-0.87	0.545	-0.545	-0.9483	-0.9483
330	1.19	0.87	0.5	-0.5	-0.87	1.0353	0.595	-0.595	-1.0353
360	1.64	1	1	0	0	1.64	1.64	0	0
	25.21					-1.7013	-1.0075	9.6422	-2.9841

$$a_0 = \frac{25.61}{6} = 4.2017$$

$a_1 = -0.28 \qquad a_2 = -0.18$

$b_1 = 1.61 \qquad b_2 = -0.5$

required Fourier Series upto Second harmonic is given by

$$y = 2.1 + (-0.28\cos x + 1.61\sin x) + (-0.18\cos 2x - 0.5\sin 2x)$$

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8th problem Solution

Interval of x is $0 \le x < 6$

$\therefore \; 2l = 6 \quad l = 3$

$N = 6 \quad \frac{2}{N} = \frac{1}{3}$

Fourier Series upto 3rd harmonic is

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right)$$
$$+ \left(a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l}\right)$$

where $l = 3$

$$y = \frac{a_0}{2} + \left(a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right)$$
$$+ \left(a_3 \cos\left(\frac{3\pi x}{3}\right) + b_3 \sin\left(\frac{3\pi x}{3}\right)\right)$$

put $\frac{\pi x}{3} = \theta$

$$y = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta)$$
$$+ (a_3 \cos 3\theta + b_3 \sin 3\theta)$$

x	$\theta = \pi x/3$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$	$y\sin\theta$	$y\sin 2\theta$	$y\sin 3\theta$
0	0	4	4	4	4	0	0	0
1	60	8	4	-4	-8	6.928	6.928	0
2	120	15	-7.5	-7.5	15	12.99	-12.99	0
3	180	7	-7	7	-7	0	0	0
4	240	6	-3	-3	6	-5.196	5.196	0
5	300°	2	1	-1	-2	-1.732	-1.732	0
		42	-8.5	-4.5	8	12.99	-2.598	0

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$$a_0 = \frac{2}{N}\sum y = \frac{1}{3}(42) = 14$$

$$a_0 = 14$$

$$a_1 = \frac{2}{N}\sum y\cos\theta = \frac{1}{3}(-8.5) = -2.833$$

$$b_1 = \frac{2}{N}\sum y\sin\theta = \frac{1}{3}\times 12.99 = 4.33$$

$$a_2 = \frac{2}{N}\sum y\cos 2\theta = \frac{1}{3}(-4.5) = -1.5$$

$$b_2 = \frac{2}{N}\sum y\sin 2\theta = \frac{1}{3}(-2.598) = -0.866$$

$$a_3 = \frac{2}{N}\sum y\cos 3\theta = \frac{1}{3}(8) = 2.667$$

$$b_3 = \frac{2}{N}\sum y\sin 3\theta = \frac{1}{3}(0) = 0$$

Required fourier series upto 3rd harmonic is

$$y = 7 - 2.833\cos\frac{\pi x}{3} + 4.33\sin\frac{\pi x}{3} - 1.5\cos\frac{2\pi x}{3}$$
$$-0.866\sin\frac{2\pi x}{3} + 2.667\cos\pi x$$

(9) obtain the constant term and first three co-efficients in the fourier cosine series for y using following table

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Soln: Here the interval of x is $0 \le x < 6$
Since co-efficients of fourier cosine series are to be found, we have to conclude that it should be cosine half range fourier series of $y = f(x)$ in $(0, 6)$

comparing with half range $(0, l)$

we get $l = 6$

$\therefore$ Fourier cosine series is of the form

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$y = \frac{a_0}{2} + a_1 \cos \frac{\pi x}{l} + a_2 \cos \frac{2\pi x}{l} + a_3 \cos \frac{3\pi x}{l} + \cdots$$

W.K.T $l = 6$

$$y = \frac{a_0}{2} + a_1 \cos \frac{\pi x}{6} + a_2 \cos \frac{2\pi x}{6} + a_3 \cos \frac{3\pi x}{6}$$

Take $\frac{\pi x}{6} = \theta$

$$y = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta \quad - (1)$$

Now, we have to find a_0 & a_1, a_2, a_3

$a_0 = \frac{2}{N} \sum y \qquad a_1 = \frac{2}{N} \sum y \cos\theta \qquad a_2 = \frac{2}{N} \sum y \cos 2\theta$

$a_3 = \frac{2}{N} \sum y \cos 3\theta$

$N = 6, \quad \frac{2}{N} = \frac{1}{3}$

x	$\theta = \frac{\pi x}{6}$	y	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	0	4	1	1	1	4	4	4
1	30	8	0.866	0.5	0	6.928	4	0
2	60	15	0.5	-0.5	-1	7.5	-7.5	-15
3	90	7	0	-1	0	0	-7	0
4	120	6	-0.5	-0.5	1	-3	-3	6
5	150	2	-0.866	0.5	0	-1.732	1	0
		42				13.696	-8.5	-5

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$\sum y = 42$ $\sum y\cos\theta = 13.696$ $\sum y\cos 2\theta = -8.5$

$\sum y\cos 3\theta = -5$

$a_0 = \frac{1}{3}(42) = 14$

$a_1 = \frac{1}{3}(13.696) = 4.565$

$a_2 = \frac{1}{3}(-8.5) = -2.833$

$a_3 = \frac{-5}{3} = -1.667$

$\therefore$ required values $\frac{a_0}{2}$, a_1, a_2, a_3 are respectively

7, $+4.565$, -2.833 & -1.667

(1) $\Rightarrow$ $y = 7 + 4.565\cos\theta + (-2.833)\cos 2\theta - 1.667\cos 3\theta$

$y = 7 + 4.565\cos\left(\frac{\pi x}{6}\right) + (-2.833)\cos\frac{2\pi x}{6} - 1.667\cos\frac{3\pi x}{6}$

Signals are converted into frequency domain i.e time domain [illegible] frequency domain [illegible]

Fourier Transforms

Infinite fourier transforms and Inverse fourier transforms

To represent the signal in frequency domain

The infinite fourier transforms or simply the fourier transform of a real valued function $f(x)$ is defined by $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{iux} dx$

main intention to convert time domain to frequency domain

on integration we obtain a f^n of u which is usually denoted by $F(u)$ or $\hat{f}(u)$

The inverse fourier transform of $F(u)$ denoted by $F^{-1}[F(u)]$ or $F^{-1}[\hat{f}(u)]$ is defined by the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) e^{-iux} du$

on integration we obtain a f^n of x

$f(x) = F^{-1}[F(u)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) e^{-iux} du$

Fourier cosine and fourier sine transforms

Inverse fourier cosine & I.F. sine transforms

If $f(x)$ is defined for all possible value of x, we define the following

TYPE	TRANSFORM	INVERSE TRANSFORM
fourier transform	$\int_{-\infty}^{\infty} f(x)e^{iux} dx = F(u)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) e^{-iux} du = f(x)$
fourier cosine transform	$\int_{0}^{\infty} f(x)\cos ux\, dx = F_c(u)$	$\frac{2}{\pi}\int_{0}^{\infty} F_c(u)\cos ux\, du = f(x)$
Fourier Sine transform	$\int_{0}^{\infty} f(x)\sin ux\, dx = F_s(u)$	$\frac{2}{\pi}\int_{0}^{\infty} F_s(u)\sin ux\, du = f(x)$

Properties

① linearity property:

$F_c[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]$

$= c_1 F_c[f_1(x)] + c_2 F_c[f_2(x)] + \cdots + c_n F_c[f_n(x)]$

② change of scale property:

If $F_c[f(x)] = F_c(u)$ then

$F_c[f(ax)] = \frac{1}{a} F_c(u/a)$

③ Modulation properties

If $F_s[f(x)] = F_s(u)$ and $F_c[f(x)] = F_c(u)$ then

① $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(u+a) + F_s(u-a)]$

② $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(u-a) - F_c(u+a)]$

③ $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(u+a) + F_c(u-a)]$

④ $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(u+a) - F_s(u-a)]$

Problems

① Find the Complex fourier transform of the function $f(x) = \begin{cases} 1 & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$

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and hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$

Soln: Fourier transform of $f(x)$ is given by

$$F(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$f(x) = \begin{cases} 1 & \text{for } -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_{-a}^{a} 1 \cdot e^{iux}\, dx$$

$$= \left[\frac{e^{iux}}{iu}\right]_{-a}^{a}$$

$$= \frac{1}{iu}\left[e^{iua} - e^{-iua}\right]$$

w.K.T $e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{1}{iu}\left[(\cancel{\cos au} + i\sin au) - (\cancel{\cos au} - i\sin au)\right]$$

$$= \frac{1}{iu}\, 2i\sin au$$

$$\boxed{F(u) = \frac{2\sin au}{u}}$$

Let us evaluate $\int_0^{\infty} \frac{\sin x}{x}\, dx$

$$F(u) = \frac{2\sin au}{u}$$

$$F(-u) = +\frac{2\sin a(-u)}{-u}$$

$$= \frac{-2\sin au}{-u} = \frac{2\sin au}{u} = F(u)$$

$F(-u) = F(u)$

$\therefore$ $F(u)$ is an even function

Inverse Fourier transform is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du = f(x)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin au}{u}\, e^{-iux}\, du = f(x)$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, e^{-iux}\, du = f(x)$$

$f(x) = 1$ for $|x| \le a$

Now, let us take $x = 0$

value of $f(x)$ at $x=0$ is 1

i.e $f(0) = 1$

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, e^{0}\, du$$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, du$$

$\frac{\sin au}{u}$ is even function

$$\pi = 2\int_{0}^{\infty} \frac{\sin au}{u}\, du$$

$$\int_{0}^{\infty} \frac{\sin au}{u}\, du = \frac{\pi}{2}$$

put $a=1$ and changing u to x

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \pi/2$$

② If $f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$

Find the fourier transform of $f(x)$ & hence find the value of

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① $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \, dx$

② $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

Sol^n:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} \, dx$$

$$f(x) = \begin{cases} 1-x^2, & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(u) = \int_{-1}^{1} (1-x^2) e^{iux} \, dx$$

Apply Bernoulli's rule

$$F(u) = \left[(1-x^2) \frac{e^{iux}}{iu} - (-2x) \frac{e^{iux}}{i^2 u^2} - 2 \frac{e^{iux}}{i^3 u^3} \right]_{-1}^{1}$$

$$= \left[(1-x^2) \frac{e^{iux}}{iu} + 2x \frac{e^{iux}}{-u^2} - \frac{2e^{iux}}{-iu^3} \right]_{-1}^{1}$$

$$i^2 = -1, \quad 1/i = -i$$

$$= \left[(1-x^2) \frac{e^{iux}}{u} \times -i - \frac{2x e^{iux}}{u^2} - \frac{i2e^{iux}}{u^3} \right]_{-1}^{1}$$

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$$= \left[\left\{0 - \frac{2e^{iu}}{u^2} - \frac{2ie^{iu}}{u^3}\right\} - \left\{0 + \frac{2e^{-iu}}{u^2} - \frac{2ie^{-iu}}{u^3}\right\}\right]$$

$$= \frac{-2e^{iu}}{u^2} - \frac{2ie^{iu}}{u^3} - \frac{2e^{-iu}}{u^2} + \frac{2ie^{-iu}}{u^3}$$

$$= \frac{-2}{u^2}(e^{iu} + e^{-iu}) - \frac{2i}{u^3}(e^{iu} - e^{-iu})$$

$$= -\frac{2}{u^2}\left[(\cos u + \cancel{i\sin u}) + (\cos u - \cancel{i\sin u})\right]$$

$$- \frac{2i}{u^3}\left[(\cancel{\cos u} + i\sin u) - (\cancel{\cos u} - i\sin u)\right]$$

$$= \frac{-2}{u^2}\left[2\cos u\right] + \left[2i\sin u\right] \times -\frac{2i}{u^3}$$

$$= \frac{-4}{u^2}\left[\cos u\right] - \frac{4(-1)\sin u}{u^3}$$

$$= \frac{-4u\cos u + 4\sin u}{u^3}$$

$$F(u) = \frac{4(\sin u - u\cos u)}{u^3}$$

Let us evaluate $\int_0^\infty \frac{-\cancel{\sin x} + x\cos x}{x^3}\,dx$

$$F(u) = \frac{4(\sin u - u\cos u)}{u^3}$$

$$F(-u) = \frac{4(-\sin u + u\cos u)}{-u^3}$$

$$= \frac{+4(\sin u - u\cos u)}{+u^3}$$

$F(-u) = F(u)$ is a even function

Inverse fourier transform is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du \quad = f(x)$$

$f(x) = 1 - x^2$

put $x = 0$, $f(0) = 1$ & use the expression of $F(u)$

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3}\, e^{0}\, du$$

$$1 = \frac{1}{2\pi} \times 8 \int_{0}^{\infty} \frac{\sin u - u\cos u}{u^3}\, du$$

$$1 = \frac{4}{\pi}\int_{0}^{\infty} \frac{\sin u - u\cos u}{u^3}\, du$$

$$\int_{0}^{\infty} \frac{\sin u - u\cos u}{u^3}\, du = +\pi/4$$

$$-\int_{0}^{\infty} \frac{u\cos u - \sin u}{u^3}\, du = \pi/4$$

$$\int_{0}^{\infty} \frac{u\cos u - \sin u}{u^3}\, du = -\pi/4$$

changing u to x

$$\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx = -\pi/4$$

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(b) $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

$$F(u) = \frac{4(\sin u - u\cos u)}{u^3}$$

$F(-u) = F(u)$, is a even function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3}\, e^{-iux}\, du$$

put $x = 1/2$

$$f(x) = 1 - x^2 = 1 - (1/2)^2 = 1 - 1/4 = 3/4$$

$$\frac{3}{4} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3}\, e^{-iu \cdot 1/2}\, du$$

$$\frac{3}{4} = \frac{4}{2\pi}\int_{-\infty}^{\infty} \frac{\sin u - u\cos u}{u^3}\left(\cos u/2 - i\sin u/2\right) du$$

$F(u) = \frac{\sin u - u\cos u}{u^3}$ is even function

$$\frac{3}{4} = \frac{\cancel{2\pi}}{\pi} \times 2\int_0^{\infty} \frac{\sin u - u\cos u}{u^3}\left(\cos u/2 - i\sin u/2\right) du$$

$$\frac{3\pi}{16} = \int_0^{\infty} \frac{(\sin u - u\cos u)}{u^3}\left(\cos u/2 - i\sin u/2\right) du$$

equating real parts on B.S

$$3\pi/16 = \int_0^{\infty} \frac{\sin u - u\cos u}{u^3} \cdot \cos u/2 \cdot du$$

Changing u to x $\int_0^{\infty} \frac{[-\sin x + x\cos x]}{x^3} \cos(x/2)\, dx = 3\pi/16$

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$$\int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\,dx = -\frac{3\pi}{16}$$

③ June Dec 2018

Find the fourier transform of $f(x) = e^{-|x|}$

$$\left\{ \begin{array}{l} |x| = x \text{ if } x \geq 0 \\ \quad\;\; -x \text{ if } x < 0 \end{array} \right.$$

Sol^n: Fourier transform of $f(x)$ is given by $F(u) = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx$

here $f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ e^{x} & \text{for } x < 0 \end{cases}$

$$F(u) = \int_{-\infty}^{0} e^{x}\cdot e^{iux}\,dx + \int_0^{\infty} e^{-x} e^{iux}\,dx$$

$$= \int_{-\infty}^{0} e^{(1+iu)x}\,dx + \int_0^{\infty} e^{-(1-iu)x}\,dx$$

$$= \left[\frac{e^{(1+iu)x}}{1+iu}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-iu)x}}{-(1-iu)}\right]_0^{\infty}$$

$$= \left\{\frac{1}{1+iu} - 0\right\} + \left\{0 - (-)\frac{1}{1-iu}\right\}$$

$$= \frac{1}{1+iu} + \frac{1}{1-iu}$$

$$= \frac{1-iu+1+iu}{(1+iu)(1-iu)}$$

$$= \frac{+2}{1+i^2u^2}$$

$$F(u) = \frac{2}{1+u^2}$$

(4) June 2018

Find the fourier transform of
$f(x) = \begin{cases} 1-|x| & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ & hence deduce

that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \pi/2$

Sol^n:

$|x| = x$ for $x \ge 0$
$= -x$ for $x < 0$

$$f(x) = \begin{cases} 1-|x| & \text{for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

$$F(u) = \int_{-1}^{1} \{1-|x|\} e^{iux}\, dx$$

$$F(u) = \int_{-1}^{0} \{1-(-x)\} e^{iux}\, dx + \int_0^1 \{1-(x)\} e^{iux}\, dx$$

$$= \int_{-1}^{0} (1+x) e^{iux}\, dx + \int_0^1 (1-x) e^{iux}\, dx$$

A.B.R

$$= \left[(1+x)\frac{e^{iux}}{iu} - (1)\frac{e^{iux}}{i^2u^2} \right]_{-1}^{0} + \left[(1-x)\frac{e^{iux}}{iu} - (-1)\frac{e^{iux}}{i^2u^2} \right]_0^1$$

$$= \left[\left\{\frac{1}{iu} + \frac{1}{u^2}\right\} - \left\{0 + \frac{e^{-iu}}{u^2}\right\}\right] + \left[\left\{0 + \frac{e^{iu}}{-u^2}\right\} - \left\{\frac{+1}{iu} + \frac{1}{-u^2}\right\}\right]$$

$$= \cancel{\frac{-i}{u}} + \frac{1}{u^2} - \frac{e^{-iu}}{u^2} - \frac{e^{iu}}{u^2} + \cancel{\frac{i}{u}} + \frac{1}{u^2}$$

$$= \frac{2}{u^2} - \frac{1}{u^2}\left[e^{-iu} + e^{iu}\right]$$

$$= \frac{2}{u^2} - \frac{1}{u^2}\left[\{\cos u - i\sin u\} + \{\cos u + i\sin u\}\right]$$

$$= \frac{2}{u^2} - \frac{2\cos u}{u^2}$$

$$= \frac{2}{u^2}(1-\cos u)$$

$$= \frac{2}{u^2} \times 2\sin^2 u/2$$

$$F(u) = \frac{4\sin^2 u/2}{u^2}$$

w.k.T

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos\theta = 2\cos^2\theta/2 - 1$

$\cos\theta = 1 - 2\sin^2\theta/2$

$2\sin^2\theta/2 = 1 - \cos\theta$

by Inverse F.T

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 u/2}{u^2}\, e^{-iux}\, du \quad \text{—} \; (*)$$

(*) >

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2 u/2}{u^2/4} \cdot e^{-iux} \, du$$

put $x=0$ $f(0)=1$

$$1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2 u/2}{(u/2)^2} \, du$$

put $u/2 = t \Rightarrow du = 2dt$

u varies from $-\infty$ to ∞

t varies from $-\infty$ to ∞

$$1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \cdot 2 \cdot dt$$

$\frac{\sin^2 t}{t^2}$ is even function

$$I = \frac{1}{\pi} 2 \int_0^{\infty} \frac{\sin^2 t}{t^2} \, dt$$

$$\int_0^{\infty} \frac{\sin^2 t}{t^2} \, dt = \frac{\pi}{2}$$

(5) Find the fourier Sine & cosine transforms of $f(x) = e^{-\alpha x}$, $\alpha > 0$

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Soln: fourier Sine and Cosine transforms are given by

$$F_S(u) = \int_0^{\infty} f(x) \sin ux \, dx \text{ and } F_C(u) = \int_0^{\infty} f(x) \cos ux \, dx$$

$$F_S(u) = \int_0^{\infty} e^{-\alpha x} \sin ux \, dx$$

w.K.T $\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx \right]$

$$= \left\{ \frac{e^{-\alpha x}}{(-\alpha)^2 + u^2} \left[-\alpha \sin ux - u \cos ux \right] \right\}_0^{\infty}$$

$e^{-\alpha x} \to 0$ as $x \to \infty$ i.e $e^{-\infty} = 0$

$e^{0} = 1$

$$= 0 - \frac{1}{\alpha^2 + u^2} \left[-\alpha \sin(0) - u \cos(0) \right]$$

$$= -\frac{1}{\alpha^2+u^2} \left[0 - u \right] = \frac{u}{\alpha^2+u^2}$$

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$$F_s(u) = \frac{u}{\alpha^2+u^2}$$

$$F_c(u) = \int_0^\infty f(x)\cos ux \, dx$$

$$= \int_0^\infty e^{-\alpha x} \cos ux \, dx$$

W.K.T $\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$= \left[\frac{e^{-\alpha x}}{(-\alpha)^2+u^2}\left(-\alpha\cos ux + u\sin ux\right)\right]_0^\infty$$

$e^{-\infty} = 0$

$$= 0 - \frac{1}{\alpha^2+u^2}\left(-\alpha\cos(0) + u\sin(0)\right)$$

$$= \frac{-1}{\alpha^2+u^2} \times -\alpha(1)$$

$$F_c(u) = \frac{\alpha}{\alpha^2+u^2}$$

Dec 2018 (6) Obtain the fourier Cosine transform of the function

$$f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4-x, & 1 < x < 4 \\ 0, & x > 4 \end{cases}$$

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Sol^n: Fourier Cosine transform is given by

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$= \int_0^1 f(x) \cos ux \, dx + \int_1^4 f(x) \cos ux \, dx + \int_4^\infty f(x) \cos ux \, dx$$

$$= \int_0^1 4x \cos ux \, dx + \int_1^4 (4-x) \cos ux \, dx + \int_4^\infty 0 \cdot \cos ux \, dx$$

$$= \int_0^1 4x \cos ux \, dx + \int_1^4 (4-x) \cos ux \, dx + 0$$

Apply B. Rule

$$= \left[4x \times \frac{\sin ux}{u} + 4 \cdot \frac{\cos ux}{u^2} \right]_0^1 + \left[(4-x) \frac{\sin ux}{u} - (-1) \times - \right.$$

$$= \left[\left\{ 4 \frac{\sin u}{u} + \frac{4 \cos u}{u^2} \right\} - \left\{ 0 + \frac{4}{u^2} \right\} \right] + \left[\left\{ 0 - \frac{\cos 4u}{u^2} \right\} - \left\{ \right. \right.$$

$$= \frac{4 \sin u}{u} - \frac{3 \sin u}{u} + \frac{4 \cos u}{u^2} + \frac{\cos u}{u^2} - \frac{4}{u^2} - \frac{\cos 4u}{u^2}$$

$$= \frac{\sin u}{u} + \frac{5 \cos u}{u^2} - \frac{1}{u^2} (4 + \cos 4u)$$

OR

$$= \frac{\sin u}{u} + \frac{5 \cos u}{u^2} - \frac{4}{u^2} - \frac{\cos 4u}{u^2}$$

OR

$$F_c(u) = \frac{\sin u}{u} + \frac{5 \cos u - 4 - \cos 4u}{u^2}$$

Do yourself

(7) Find the fourier cosine transform of

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$$

Sol^n:

$$F_c(u) = \int_0^{\infty} f(x)\cos ux \, dx$$

$$= \int_0^1 f(x)\cos ux\,dx + \int_1^2 f(x)\cos ux\,dx + \int_2^{\infty} f(x)\cos ux\,dx$$

$$= \int_0^1 x\cos ux\,dx + \int_1^2 (2-x)\cos ux\,dx + \int_2^{\infty} 0\cdot\cos ux\,dx$$

$$\left.\frac{\cos ux}{u^2}\right]_1^4$$

$$= \int_0^1 x\cos ux\,dx + \int_1^2 (2-x)\cos ux\,dx + 0$$

$$\frac{3\sin u}{u} - \frac{\cos u}{u^2}\Big\}\Big]$$

$$= \left[x\frac{\sin ux}{u} + \frac{\cos ux}{u^2}\right]_0^1 + \left[(2-x)\frac{\sin ux}{u} - (-1)\times -\frac{\cos ux}{u^2}\right]_1^2$$

$$= \left\{\left(\frac{\sin u}{u} + \frac{\cos u}{u^2}\right) - \left(0 + \frac{1}{u^2}\right)\right\} + \left\{\left(0 - \frac{\cos 2u}{u^2}\right) - \left(\frac{\sin u}{u} - \frac{\cos u}{u^2}\right)\right\}$$

$$= \cancel{\frac{\sin u}{u}} + \frac{\cos u}{u^2} - \frac{1}{u^2} - \frac{\cos 2u}{u^2} - \cancel{\frac{\sin u}{u}} + \frac{\cos u}{u^2}$$

$$= \frac{2\cos u}{u^2} - \frac{1}{u^2} - \frac{\cos 2u}{u^2}$$

$$F_c(u) = \frac{2\cos u - \cos 2u - 1}{u^2}$$

⑧ Find the fourier Sine transform of $f(x) = e^{-|x|}$ & hence evaluate

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$$\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx \quad m>0$$

Sol: Fourier Sine transform is given by

$$F_S(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$F_S(u) = \int_0^\infty e^{-|x|} \sin ux \, dx$$

$$= \int_0^\infty e^{-x} \sin ux \, dx \quad \text{Since } |x| = x,\ x>0$$

W.K.T $\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

$$= \left[\frac{e^{-x}}{(-1)^2 + u^2} \left\{ -\sin ux - u \cos ux \right\} \right]_0^\infty$$

$$= \left[\frac{e^{-x}}{1^2+u^2} (-\sin ux - u \cos ux) \right]_0^\infty$$

$\Rightarrow$ $e^{-\infty} = 0$, $e^0 = 1$, $\sin 0 = 0$, $\cos 0 = 1$

$$= 0 - \frac{1}{1+u^2} (0 - u)$$

$$F_S(u) = \frac{u}{1+u^2}$$

By inverse fourier sine transform we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u) \sin ux \, du$$

put $x = m$ $\quad f(x) = e^{-|m|} = e^{-m}$

$$e^{-m} \cdot \frac{\pi}{2} = \int_0^\infty \frac{u}{1+u^2} \sin um \cdot du$$

$$\int_0^\infty \frac{u \sin mu}{1+u^2} du = \frac{\pi}{2} \cdot e^{-m}$$

changing u to x

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$

(9) Find the fourier sine transform of $\frac{e^{-ax}}{x}$, $a > 0$

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Sol^n: $F_s(u) = \int_0^\infty f(x) \sin ux \, dx$

$$F_s(u) = \int_0^\infty \frac{e^{-ax}}{x} \sin ux \, dx \quad \text{--- (1)}$$

we cannot evaluate this integral directly and hence we proceed as follows

$$\frac{d}{du}[F_s(u)] = \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial u}(\sin ux) \, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cdot \cos ux \cdot \cancel{x} \, dx$$

$$F_S(u) = \int_0^\infty e^{-ax} \cos ux \, dx \quad — (*)$$

$$= \left[\frac{e^{-ax}}{(-a)^2+u^2} (-a\cos ux + u \sin ux) \right]_{x=0}^{\infty}$$

$$= 0 - \frac{1}{a^2+u^2}(-a+0)$$

$$\frac{d}{du}\left[F_S(u)\right] = \frac{a}{a^2+u^2}$$

by integrating w.r.to u on B.S we get

$$\int \frac{d}{du} F_S(u) \cdot du = \int \frac{a}{a^2+u^2} du$$

$$F_S(u) = \tan^{-1}(u/a) + C$$

To find C, put $u=0$

$$F_S(0) = \tan^{-1}(0) + C$$

$F_S(0) = 0$ from (1)

$$\underline{F_S(u) = \tan^{-1} u/a}$$

(10) Dec 2016 Find the inverse Fourier sine transform of $\hat{f}_S(\alpha) = \frac{1}{\alpha} e^{-a\alpha}, \ a > 0$

By data $\hat{f}_S(\alpha) = \frac{1}{\alpha} e^{-a\alpha}$

Sol^n:

$$F_S[f(x)] = \hat{f}_S(\alpha) = \int_0^\infty f(x) \sin \alpha x \, dx$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \hat{f}_S(\alpha) \sin \alpha x \, d\alpha \quad — (1)$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{e^{-\alpha a}}{\alpha} \sin \alpha x \, d\alpha$$

D. w. r. to x

$$\frac{d}{dx}[f(x)] = \frac{2}{\pi} \int_0^{\infty} \frac{d}{d} \left(\frac{e^{-\alpha a}}{\alpha} \frac{\partial}{\partial x}(\sin \alpha x) \right) d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} \left[\frac{e^{-\alpha a}}{\cancel{\alpha}} \times \cos \alpha x \cdot \cancel{\alpha} \right] d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-\alpha a} \cos \alpha x \cdot d\alpha$$

$$= \frac{2}{\pi} \left[\frac{e^{-a\alpha}}{a^2 + x^2} (-a \cos \alpha x + x \sin \alpha x) \right]_0^{\infty}$$

$$= \frac{2}{\pi} \left[0 - \frac{1}{a^2+x^2} (-a + 0) \right]$$

$$\frac{d}{dx} f(x) = \frac{2}{\pi} \times \frac{a}{a^2+x^2}$$

Int. w. r. to x

$$f(x) = \frac{2}{\pi} \int_0 \frac{a}{a^2+x^2} dx$$

$\tan^{-1} x/a = \frac{1}{1 + x^2/a^2}$

$=$

$$f(x) = \frac{2}{\pi} \tan^{-1} \frac{x}{a} + C \quad \text{---} \quad (*)$$

To find C, put $x = 0$ in $(*)$

$$f(0) = 0 + C$$

$$0 = 0 + C \Rightarrow \underline{C = 0}$$

$$f(x) = \frac{2}{\pi} \tan^{-1} x/a$$

(*) find the fourier Cosine transform of

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$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$$

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(17) Find the infinite fourier Cosine transform of e^{-x^2}

Sol: Fourier cosine transform is given by

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$F_c(u) = \int_0^\infty e^{-x^2} \cos ux \, dx$$

we cannot evaluate the integral directly & hence proceed as follows. The process is called diffⁿ under the integral sign.

$$\frac{dF_c}{du} = \int_0^\infty \frac{\partial}{\partial u}\left(e^{-x^2} \cos ux\right) dx$$

$$= \int_0^\infty e^{-x^2}(-\sin ux \cdot x)\, dx$$

$$= \frac{1}{2}\int_0^\infty \sin ux \left\{e^{-x^2}(-2x)\right\} dx$$

$$2\frac{dF_c}{du} = \int_0^\infty \sin ux \left\{e^{-x^2}(-2x)\right\} dx$$

Integrate RHS by parts we have

$$= \left[\sin ux \left(e^{-x^2}\right)\right]_0^\infty - \int_0^\infty e^{-x^2}(\cos ux \cdot u)\, dx$$

MODULE - 04

Numerical Solution of ordinary differential equations of first order and first degree

Numerical methods for initial Value problems

Consider a differential equation of first order and first degree in the form

$\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$

i.e $y = y_0$ at $x = x_0$

This problem of finding y is called an initial Value problem.

Taylor's Series method

Consider the initial value problem

$\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$

The solution $y(x)$ is approximated to a power series in $x - x_0$ using Taylor's Series. Then we can find the Value of y for Various Values of x in the neighbourhood of x_0.

we have Taylor's Series expansion $y(x)$ about the point x_0 in the form:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \ldots$$

Here $y'(x_0)$, $y''(x_0)$ .... denote the Value of the derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... at x_0 which Can be found by making use of the data

Problem

① (a) Use Taylor's series method to find y at $x = 0.1, 0.2, 0.3$ considering terms upto the third degree given that $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$

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Sol^n:

Taylor's series expansion of $y(x)$ is given by

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \dots$$

By data

$y(x_0) = y_0 \Rightarrow x_0 = 0 \quad y_0 = 1$ and $y' = x^2 + y^2$

$y(0) = 1$

$$\therefore y(x) = y(0) + (x-0) y'(0) + \frac{(x-0)^2}{2} y''(0) + \frac{(x-0)^3}{6} y'''(0)$$

$$y(x) = y(0) + x y'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0) \quad \text{—— ①}$$

we need to compute $y'(0), y''(0), y'''(0)$

Consider $y' = x^2 + y^2$ —— ②

we initial value $y'(0) = 0^2 + 1$

$$\boxed{y'(0) = 1}$$

Differentiating ② w.r.to x we have

$$y'' = 2x + 2yy' \quad \text{—— ③}$$

$$y''(0) = 2(0) + 2y(0) y'(0)$$

$$= 0 + 2(1)(1)$$

$$\boxed{y''(0) = 2}$$

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Differentiating (3) w. r. to x

$$y''' = 2 + 2[y y'' + y' \cdot y']$$

$$y''' = 2 + 2[y y'' + (y')^2]$$

$$y'''(0) = 2 + 2[y(0) y''(0) + [y'(0)]^2]$$

$$y'''(0) = 2 + 2[(1)(2) + 1^2]$$

$$y'''(0) = 2 + 2[2+1]$$

$$\boxed{y'''(0) = 8}$$

$y'''(x_0) + \ldots$

Substituting these values in (1) we have

$$y(x) = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{6}(8)$$

$$y(x) = 1 + x + x^2 + 4\frac{x^3}{3} \quad \text{---} \; (*)$$

This is called a Taylor's Series approximatio upto third degree and we need to put $x = 0.1, 0.2, 0.3$ in (*) we have

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3} = 1.1113$$

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3} = 1.2506$$

$$y(0.3) = 1 + 0.3 + (0.3)^2 + \frac{4(0.3)^2}{3} = 1.426$$

(2) Find y at $x = 1.02$ correct to five decimal places given $dy = (xy - 1)\,dx$ and $y = 2$ at $x=1$ applying Taylor's Series method.

Solⁿ:

Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

By data

$y(x_0) = y_0$, $y(1) = 2$, $x_0 = 1$ $y_0 = 2$ and $y' = \frac{dy}{dx} = xy - 1$

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2}y''(1) + \frac{(x-1)^3}{6}y'''(1) \text{ — } ①$$

Consider $y' = xy - 1$ — ②

$y'(1) = (1)y(1) - 1$

use initial value

$= (1)(2) - 1$

$\boxed{y'(1) = 1}$

Differentiate ① w.r. to x

$y'' = xy' + y(1)$

$y'' = xy' + y$ — ③

use initial value and $y'(1)$ value

$y''(1) = (1)y'(1) + y(1)$

$y''(1) = (1)(1) + 2$

$\boxed{y''(1) = 3}$

Differentiate ③ w.r. to x

$y''' = xy'' + y'(1) + y'$

$y''' = xy'' + 2y'$

$$y'''(1) = (1)\,y''(1) + 2y'(1)$$

$$y'''(1) = (1)(3) + 2(1)$$

$$\boxed{y'''(1) = 5}$$

we need to find $y(1\cdot02)$
put $x = 1\cdot02$ in ①

$$y(1\cdot02) = 2 + (1\cdot02 - 1)(1) + \frac{(1\cdot02 - 1)^2}{2}(3) + \frac{(1\cdot02 - 1)^3}{6} \times 5$$

$$= 2 + 0\cdot02 + \frac{(0\cdot02)^2}{2} \times 3 + \frac{(0\cdot02)^3}{6} \times 5$$

$$= 2 + 0\cdot02 + 0\cdot0006 + 0\cdot000006$$

$$y(1\cdot02) = 2\cdot020606$$

③ From Taylor's Series method, find $y(0\cdot1)$ Considering upto fourth degree term if $y(x)$ satisfies the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$

Solⁿ: Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{4}(x_0) + \cdots$$

By data $x_0 = 0$, $y_0 = 1$ i.e $y(0) = 1$

$$y' = x - y^2$$

$$y(x) = y(0) + (x-0)y'(0) + \frac{(x-0)^2}{2!}y''(0) + \frac{(x-0)^3}{3!}y'''(0) + \frac{(x-0)^4}{4!}y^4(0)$$

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) + \frac{x^4}{24}y^4(0) \quad \text{—— ①}$$

Consider $y' = x - y^2$ —② , $y'(0) = 0 - [y(0)]^2$

use initial values.

$$y'(0) = 0 - 1^2 = -1$$

$$\therefore \boxed{y'(0) = -1}$$

Differentiate ② w. r. to x

use initial values & y'(0) value

$y'' = 1 - 2yy'$ —③; $y''(0) = 1 - 2y(0)y'(0)$

$$y''(0) = 1 - 2(1)(-1)$$

$$y''(0) = 1 + 2$$

$$\boxed{y''(0) = 3}$$

Diff. ③ w. r. to x

$$y''' = 0 - 2[yy'' + y'y']$$

$$y''' = -2[yy'' + (y')^2] \quad \text{—— ④}$$

$$y'''(0) = -2[y(0)y''(0) + \{y'(0)\}^2]$$

$$= -2[(1)(3) + (-1)^2]$$

$$y'''(0) = -2[3+1]$$

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$-0)^4 y^4(0)$
$4!$

$$\boxed{y'''(0) = -8}$$

Diff. ④ w.r. to x

$-①$

$$y^{(4)} = -2\left[y y''' + y''y' + 2y'y''\right]$$
$$= -2\left[y y''' + 3y'y''\right]$$
$$y^{(4)}(0) = -2\left[y(0)\,y'''(0) + 3\,y'(0)\,y''(0)\right]$$

$$= -2\left[(1)(-8) + 3(-1)(3)\right]$$
$$= -2\left[-8 - 9\right]$$
$$= -2 \times -17$$

$$\boxed{y^{(4)}(0) = 34}$$

Substitute all these values in ①

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(3) + \frac{x^3}{6}(-8) + \frac{x^4}{\cancel{24}_{12}} \times \cancel{34}^{17}$$

$$y(x) = 1 - x + \frac{3x^2}{2} - \frac{8x^3}{6} + \frac{17x^4}{12} \quad \text{---} \; ⊛$$

we need to find $y(0.1)$
put $x = 0.1$ in ⊛

$$y(0.1) = 1 - (0.1) + \frac{3(0.1)^2}{2} - \frac{8(0.1)^3}{6} + \frac{17(0.1)^4}{12}$$

$$= 1 - 0.1 + 0.015 - 0.00136 + 0.00014$$

$$\boxed{y(0.1) = 0.91384}$$

(H) Dec 2017 Use Taylor's Series method to find $y(4.1)$ given that $\frac{dy}{dx} = \frac{1}{x^2+y}$ and $y(4) = 4$

Soln: Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

By data $y(x_0) = y_0$

$y(4) = 4$

i.e $x_0 = 4 \quad y_0 = 4 \quad , \; y' = \frac{1}{x^2+y}$

$$y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2}y''(4) + \frac{(x-4)^3}{6}y'''(4) - $$

Consider $y' = \frac{1}{x^2+y}$

$y'(x^2+y) = 1$ — (2)

Substituting the initial values

$y'(4)[4^2 + y(4)] = 1$

$y'(4)[16+4] = 1 \Rightarrow y'(4)[20] = 1$

$\Rightarrow y'(4) = \frac{1}{20} = 0.05 \Rightarrow \boxed{y'(4) = 0.05}$

Diff. (2) w.r. to x

$y'[2x + y'] + (x^2+y)y'' = 0$ — (3)

Substituting initial values and value of $y'(4)$

$y'(4)[2(4) + y'(4)] + [4^2 + y(4)]y''(4) = 0$

$0.05[8 + 0.05] + [16+4]y''(4) = 0$

$0.4025 + 20y''(4) = 0$

$$20y''(4) = -0.4025$$

$$y''(4) = \frac{-0.4025}{20}$$

$$\boxed{y''(4) = -0.020125}$$

3rd degree term can be neglected ∵ values of derivatives are very small

Diff. (5) w.r.t. to x

$$y'[2 + y''] + [2x + y']y'' + [x^2 + y]y''' + y''[2x + y'] = 0 \quad -(1)$$

$$2y' + y'y'' + 2xy'' + y'y'' + x^2y''' + yy''' + 2xy'' + y'y'' = 0$$

$$4xy'' + 3y'y'' + 2y' + x^2y''' + yy''' = 0$$

$$4(4)y''(4) + 3y'(4)y''(4) + 2y'(4) + (4)^2y'''(4) + y(4)y'''(4) = 0$$

$$16(-0.020125) + 3(0.05)(-0.020125) + 2(0.05) + 16y'''(4) + (4)y'''(4) = 0$$

$$-0.22502 + 20y'''(4) = 0$$

$$20y'''(4) = 0.22502$$

$$\boxed{y'''(4) = +0.01125}$$

put all these values in (1)

$$y(x) = 4 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.020125) + \frac{(x-4)^3}{6}(-0.01125) \quad -(*)$$

we need to find $y(4.1)$

put $x = 4.1$ in (*)

$$y(4.1) = 4 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.020125) + \frac{(4.1-4)^3}{6}(-0.01125)$$

$y(4.1) = 4 + 0.005 - 0.0001 - 0.0000018$

$\boxed{y(4.1) = 4.00489}$

5) a) Employ Taylor's Series method to find y at $x = 0.1$ and 0.2 correct to four places of decimal in step size of 0.1 given the linear differential equation $\frac{dy}{dx} - 2y = 3e^{x}$ whose solution passes through the origin. Also find $y(0.1)$ and $y(0.2)$ by analytical method

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b) Compute $y(0.1)$

Soln:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \dots$$

origin (0,0)

By data $y(0) = 0$

i.e. $x_0 = 0 \quad y_0 = 0$, $y' - 2y = 3e^{x}$

$y' = 3e^{x} + 2y$

$$y(x) = y(0) + (x - 0)y'(0) + \frac{(x - 0)^2}{2}y''(0) + \frac{(x - 0)^3}{6}y'''(0)$$

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \quad \text{--- (1)}$$

Step (1): we need to find $y(0.1)$

Consider $y' = 3e^{x} + 2y$ --- (2)

we initial conditions

$y'(0) = 3e^{0} + 2y(0)$

$y'(0) = 3(1) + 2(0)$

$\boxed{y'(0) = 3}$

Diff. ② or y' w.r. to x

$y'' = 3e^x + 2y'$ —— ③

use initial conditions & value of $y'(0)$

$y''(0) = 3e^0 + 2y'(0)$

$y''(0) = 3(1) + 2(3)$

$y''(0) = 9$

Diff. ③ w.r. to x

$y''' = 3e^x + 2y''$

$y'''(0) = 3e^0 + 2y''(0)$

$y'''(0) = 3(1) + 2(9)$

$y'''(0) = 21$

we need to find $y(0.1)$

Now eqn ① becomes

$$y(x) = 0 + 3x + \frac{x^2}{2}(9) + \frac{x^3}{6}(21) \quad — \quad (*)$$

put $x = 0.1$ in $(*)$

$$y(0.1) = 0 + 3(0.1) + \frac{(0.1)^2(9)}{2} + \frac{(0.1)^3(21)}{6}$$

$$y(0.1) = 0.3 + 0.045 + 0.0035$$

$$\underline{\underline{y(0.1) = 0.3485}}$$

Step ②: we shall find $y(0.2)$

$y(0.1) = 0.3485$

i.e $x_0 = 0.1$ $y_0 = 0.3485$

Taylors Series expansion is given by

① ... ... ①

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \ldots$$

$$y(x) = y(0.1) + (x-0.1)y'(0.1) + \frac{(x-0.1)^2}{2}y''(0.1) + \frac{(x-0.1)^3}{6}y'''$$ — (4)

Consider $y' = 3e^x + 2y$ — (5)

$y'(0.1) = 3e^{0.1} + 2y(0.1) \Rightarrow y'(0.1) = 3e^{0.1} + 2(0.3485)$

$y'(0.1) = 4.0125$ //

Diff (5) w.r. to x

$y'' = 3e^x + 2y'$ — (6)

$y''(0.1) = 3e^{0.1} + 2y'(0.1) \Rightarrow y''(0.1) = 3e^{0.1} + 2(4.0125)$

$y''(0.1) = 11.3405$ //

Diff (6) w.r. to x

$y''' = 3e^x + 2y''$

$y'''(0.1) = 3e^{0.1} + 2y''(0.1) \Rightarrow y'''(0.1) = 3e^{0.1} + 2(11.3405)$

$y'''(0.1) = 25.9965$ //

put all these values in (4)

$$y(x) = 0.3485 + (x-0.1)(4.0125) + \frac{(x-0.1)^2}{2}(11.3405) + \frac{(x-0.1)^3}{6}25.9965$$

we need to find $y(0.2)$, So put $x = 0.2$

$$y(0.2) = 0.3485 + (0.2-0.1)(4.0125) + \frac{(0.2-0.1)^2}{2}(11.3405) + \frac{(0.2-0.1)^3}{6} \times 25.9965$$

$$y(0.2) = 0.3485 + (0.1)(4.0125) + \frac{(0.1)^2}{2}(11.3405)$$

$$+ \frac{(0.1)^3}{6}(25.9965)$$

$$= 0.3485 + 0.40125 + 0.0567025 + 0.004332$$

$$= 0.81078$$

$y(0.2)$ is 0.8108

Thus

Solution by analytic method

$\frac{dy}{dx} - 2y = 3e^x$ is of the form $\frac{dy}{dx} + Py = Q$

where $P = -2$, $Q = 3e^x$

Solution : $ye^{\int P dx} = \int Q e^{\int P dx} dx + C$

$$ye^{\int -2dx} = \int 3e^x e^{\int -2dx} dx + C$$

$$ye^{-2x} = \int 3e^x e^{-2x} dx + C$$

$$ye^{-2x} = \int 3e^{-x} dx + C$$

$$ye^{-2x} = 3\frac{e^{-x}}{-1} + C$$

$$ye^{-2x} = -3e^{-x} + C$$

$$y = \frac{-3e^{-x}}{e^{-2x}} + \frac{C}{e^{-2x}}$$

$$y = -3e^{-x} \cdot e^{2x} + ce^{2x}$$

$$y = -3e^{-x+2x} + ce^{2x}$$

$y = -3e^x + ce^{2x}$ is general solution

apply initial condition $y(0) = 0$ in the above equation

$$y(0) = -3e^0 + ce^0$$

$$0 = -3 + C$$

$$C = 3$$

put $C = 3$ in $y = -3e^{x} + Ce^{2x}$

$y = -3e^{x} + 3e^{2x}$

$y = 3(e^{2x} - e^{x})$ is the solution

Let us find $y(0.1)$ and $y(0.2)$ so

put $x = 0.1$ in the above soln $y(0.1) = 3(e^{2(0.1)} - e^{0.1})$

$y(0.1) = 0.34869$

$y(0.1) = 0.3487$

put $x = 0.2$ in the above solution

$y(0.2) = 3(e^{2(0.2)} - e^{0.2})$

$= 0.81126$

$y(0.2) = 0.8113$ by analytical method

(6) Using Taylor's Series method, obtain the value of y at $x = 0.1, 0.2, 0.3$ to four significant figures if y satisfies the equation $y'' = -xy$ given that $y' = 0.5$ and $y = 1$ when $x = 0$ taking the first five terms of the Taylor's Series expansion

Soln: Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) +$$

By data $x_0 = 0$, $y_0 = 1$

$y' = 0.5$

$y'' = -xy$

put $x_0 = 0$ in above formula

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \quad \text{—— ①}$$

consider $y'' = -xy$ —— ②

$y''(0) = -(0)(1) = 0 \Rightarrow y''(0) = 0$

eqn ② D.w.r. to x

$y''' = -[xy' + y(1)]$

$y''' = -xy' - y$ —— ③

$y'''(0) = -(0)\,y'(0) - y(0)$

$y'''(0) = -(0)(0.5) - 1$

$y'''(0) = -1$

put all these values in ①

$$y(x) = 1 + x(0.5) + \frac{x^2}{2}(0) + \frac{x^3}{6}(-1) \quad \text{—— (*)}$$

Now we need to find $y(0.1)$, $y(0.2)$ & $y(0.3)$

$$y(0.1) = 1 + (0.1)(0.5) + \frac{(0.1)^2}{2}(0) + \frac{(0.1)^3}{6}(-1)$$

$y(0.1) = 1.0498$

put $x = 0.2$ in (*)

$$y(0.2) = 1 + (0.2)(0.5) + \frac{(0.2)^2}{2}(0) + \frac{(0.2)^3}{6}(-1)$$

$y(0.2) = 1.0986$

$$y(0.3) = 1 + (0.3)(0.5) - \frac{(0.3)^2}{2}(0) + \frac{(0.3)^3}{6} \times -1$$

$$= 1 + 0.15 - 0 - 0.0045$$

$$= \underline{1.1455}$$

$\therefore y(0.1) = 1.00498$, $y(0.2) = 1.0986$ $y(0.3) = 1.1455$

(7) Use Taylor's Series method to solve $y' = x^2 + y$ in the range $0 \le x \le 0.2$ by taking stepsize $h = 0.1$ given that $y = 10$ at $x = 0$ initially considering terms upto the fourth degree.

Solⁿ:

In this problem
Since the step size is specified as 0.1, the problem has to be done in two stages we have to first find $y(0.1)$ and use this as the initial condition to find $y(0.2)$
Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0)$$

$$+ \frac{(x-x_0)^4}{4!} y^{4}(x_0) + \ldots \quad (*)$$

I stage: By data

$$y' = x^2 + y, \quad x_0 = 0 \quad y_0 = 10$$

$$y'(0) = 0^2 + y(0) \quad \text{use initial conditions}$$

$$y'(0) = 0 + 10$$

$$y'(0) = 10$$

Differentiating y' w. r. to x

$y'' = 2x + y'$

$y''(0) = 2(0) + y'(0) \Rightarrow y''(0) = 0 + 10 \Rightarrow y''(0) = 10$//

Diff. y'' w. r. to x

$y''' = 2 + y''$

$y'''(0) = 2 + y''(0) \Rightarrow y'''(0) = 2 + (10) \Rightarrow y'''(0) = 12$//

D. y''' w. r. to x

$y^4 = 0 + y'''$

$y^4(0) = 0 + y'''(0) \Rightarrow y^4(0) = 12$//

Now we have to find y at $x = 0.1$

So put $x = 0.1$ with $x_0 = 0$

$\therefore$ (*) becomes

$$y(0.1) = y(0) + (0.1-0)y'(0) + \frac{(0.1-0)^2}{2} y''(0) + \frac{(0.1-0)^3}{6} y'''(0) + \frac{(0.1-0)^4}{24} y^4(0)$$

$$y(0.1) = 10 + (0.1)(10) + \frac{(0.1)^2}{2}(10) + \frac{(0.1)^3}{6}(12) + \frac{(0.1)^4}{24}(12)$$

$$= 10 + 1 + 0.05 + 0.002 + 0.00005$$

$$y(0.1) = \underline{11.05205} \approx \underline{11.052}$$

II Stage: Now take $x_0 = 0.1$, $y_0 = 11.052$

we have

$y' = x^2 + y$

we initial conditions

$y'(0.1) = (0.1)^2 + 11.052 = 11.062 \Rightarrow y'(0.1) = 11.062$ //

Diff y' w.r. to x

$y'' = 2x + y'$;

$y''(0.1) = 2(0.1) + y'(0.1) \Rightarrow y''(0.1) = 0.2 + 11.062$

$y''(0.1) = 11.262$ //

Diff. y'' w.r. to x

$y''' = 2 + y''$

$y'''(0.1) = 2 + y''(0.1) \Rightarrow y'''(0.1) = 2 + 11.262$

$y'''(0.1) = 13.262$ //

Diff y''' w.r. to x

$y^4 = 0 + y'''$

$y^4(0.1) = y'''(0.1) \Rightarrow y^4(0.1) = 13.262$ //

Now we have to find $y(0.2)$

[i.e y at $x = 0.2$]

So put $x = 0.2$ with $x_0 = 0.1$

(*) becomes

$$y(0.2) = y(0.1) + (0.2-0.1)y'(0.1) + \frac{(0.2-0.1)^2}{2}y''(0.1)$$

$$+ \frac{(0.2-0.1)^3}{6}y'''(0.1) + \frac{(0.2-0.1)^4}{4!}y^4(0.1)$$

$$= 11.052 + (0.1)11.062 + \frac{(0.1)^2}{2}(11.262) + \frac{(0.1)^3}{6}(13.26$$

$$+ \frac{(0.1)^4}{24} \times 13.262$$

$= 11.052 + 1.1062 + 0.05631 + 0.00221 + 0.000055$

$y(0.2) = 12.21677$

Thus $y(0.1) = 11.052$ and $y(0.2) = 12.21677$

(8) Use Taylor's Series method to obtain a power series in $(x-4)$ for the equation $5x\frac{dy}{dx} + y^2 - 2 = 0$, $x_0 = 4$, $y_0 = 1$ and use it to find y at $x = 4.1, 4.2, 4.3$ correct to four decimal places.

Soln: Taylor's Series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

By data $x_0 = 4$, $y_0 = 1$

So

$$y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2}y''(4) + \frac{(x-4)^3}{6}y'''(4) \quad \text{—(1)}$$

Consider $5xy' + y^2 - 2 = 0$ —— (2)

Substitute initial values we obtain

[NOTE: $y' = y'(x)$]

$$5(4)y'(4) + (y(4))^2 - 2 = 0$$

$$20y'(4) + (1)^2 - 2 = 0$$

$$20y'(4) = 1$$

$$y'(4) = 1/20 = 0.05$$

$$y'(4) = 0.05$$

Diff. ② w.r. to x

$$5[xy'' + y' \times 1] + [2yy' - 0] = 0$$

$$5[xy'' + y'] + [2yy'] = 0 \quad \text{---} ③$$

use initial condition

$$5[(4)y''(4) + y'(4)] + [2y(4)y'(4)] = 0$$

$$5[4y''(4) + 0.05] + [2(1)(0.05)] = 0$$

$$20y''(4) + 0.25 + 0.1 = 0$$

$$20y''(4) + 0.35 = 0$$

$$20y''(4) = -0.35$$

$$y''(4) = -0.0175$$

Since the value of the second derivative itself y small enough we shall approximate up to second degree only

Sub. all value in ①

$$y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.0175) \quad \text{---} ⊛$$

Now we have to find y at $x = 4.1, 4.2, 4.3$

So, put $x = 4.1$ in ⊛

$$y(4.1) = 1 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.0175)$$

$$y(4.1) = 1.0049$$

put $x = 4.2$ in ⊛

$$y(4.2) = 1 + (4.2-4)(0.05) + \frac{(4.2-4)^2}{2}(-0.0175)$$

$y(4.2) = 1.0097$

put $x = 4.3$ in (*)

$y(4.3) = 1 + (4.3-4)(0.05) + \frac{(4.3-4)^2}{2}(-0.0175)$

$y(4.3) = 1.0142$

(OR)

Diff (2) w.r. to x — If you consider derivatives upto 3rd degree answer will not changes

$5[x y''' + y'' + y''] + 2[y y'' + y' y'] = 0$

$5[x y''' + 2y''] + 2[y y'' + (y')^2] = 0$

$5[(4) y'''(4) + 2y''(4)] + 2[y(4) y''(4) + (y'(4))^2] = 0$

$5[4 y'''(4) + 2(-0.0175)] + 2[(1)(-0.0175) + (0.05)^2] = 0$

$20 y'''(4) - 0.175 + (-0.035) + 0.005 = 0$

$20 y'''(4) - 0.205 = 0$

$y'''(4) = \frac{0.205}{20}$

$y'''(4) = 0.01025$

Put all these in (1)

$y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.0175) + \frac{(x-4)^3}{6}(0.01025)$

we need to find y at $x = 4.1, 4.2, 4.3$

$y(x) - y(4.1) = 1.0049$

$y(4.2) = 1.0097$

$y(4.3) = 1.0142$

Modified Euler's Method

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \quad ; \quad y(x_0) = y_0$$

We need to find y at $x_1 = x_0 + h$
we first obtain $y(x_1) = y_1$ by applying Euler's formula and this value is regarded as the initial approximation for y_1 usually denoted by $y_1^{(0)}$ also called as predicted value of y_1.

Euler's formula is given by

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

Since the accuracy is poor in this formula this value y_1 is successively corrected to the desired degree of accuracy by the following modified Euler's formula where the successive approximations are denoted by $y_1^{(1)}, y_1^{(2)}, y_1^{(3)} \ldots$ etc

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

Euler's formula and ~~Euler's~~ modified Euler's formula jointly called as Euler's predictor and corrector formula

Problems

① ⓐ Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y = 2$ at $x = 1$ find the approximate value of y at $x = 1.4$ by taking step size $h = 0.2$ applying modified Euler's method. Also find the value of y at $x = 1.2$ and 1.4 from the analytical solution of the equation

Dec 2018

ⓑ find $y(1.2)$ in two steps.

Soln: The problem has to be done in 2 stages

I Stage:

$x_0 = 1$, $y_0 = 2$ $\frac{dy}{dx} = f(x, y) = 1 + \frac{y}{x}$ $h = 0.2$

$x_1 = x_0 + h = 1 + 0.2 = 1.2 \Rightarrow x_1 = 1.2$

$y(x_1) = y_1 = y(1.2) = ?$

Now

$f(x_0, y_0) = 1 + \frac{y_0}{x_0} = 1 + \frac{2}{1} = 1 + 2 = 3$

$f(x_0, y_0) = 3$

we have Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 2 + 0.2 (3)$$
$$= 2.6$$

we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 2 + \frac{0.2}{2}\left[3 + \left\{1 + \frac{y_1^{(0)}}{x_1}\right\}\right]$$
$$= 2 + 0.1\left[3 + 1 + \frac{2.6}{1.2}\right]$$

$$y_1^{(1)} = 2.61666$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 2 + \frac{0.2}{2}\left[3 + \left\{1 + \frac{y_1^{(1)}}{x_1}\right\}\right]$$

$$= 2 + 0.1\left[3 + 1 + \frac{2.61666}{1.2}\right]$$

$$y_1^{(2)} = 2.61805$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 2 + \frac{0.1}{2}\left[3 + \left\{1 + \frac{y_1^{(2)}}{x_1}\right\}\right]$$

$$= 2 + 0.1\left[3 + 1 + \frac{2.61805}{1.2}\right]$$

$$= 2.61817$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 2 + \frac{0.2}{2}\left[3 + \left\{1 + \frac{y_1^{(3)}}{x_1}\right\}\right]$$

$$= 2 + 0.1\left[3 + 1 + \frac{2.61817}{1.2}\right]$$

$$= 2.61818$$

$$\therefore \quad y(1.2) = 2.61818$$

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Required solution for (b) is

$y_1^{(2)} = y(1.2) = 2.61805$

II stage: we repeat the process by taking $y(1.2) \doteq 2.61818$ as the initial condition

$x_0 = 1.2 \quad y_0 = 2.61818$

$f(x_0, y_0) = 1 + \frac{y_0}{x_0} = 1 + \frac{2.61818}{1.2} = 3.1818$

$f(x_0, y_0) = 3.1818, \quad x_1 = x_0 + h \Rightarrow x_1 = 1.4, \quad y(x_1) = y_1 = y(1.4) = ?$

we have Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 2.61818 + (0.2)(3.1818)$$
$$= 3.25454$$

Now we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 2.61818 + \frac{0.2}{2}\left[3.1818 + 1 + \frac{y_1^{(0)}}{x_1}\right]$$
$$= 2.61818 + 0.1\left[3.1818 + 1 + \frac{3.25454}{1.4}\right]$$
$$= 3.2688$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 2.61818 + \frac{0.2}{2}\left[3.1818 + 1 + \frac{y_1^{(1)}}{x_1}\right]$$
$$= 2.61818 + 0.1\left[3.1818 + 1 + \frac{3.2688}{1.4}\right]$$
$$= 3.2698$$

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$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$

$= 2.61818 + \frac{0.2}{2}\left[3.1818 + \left\{1 + \frac{y_1^{(2)}}{x_1}\right\}\right]$

$= 2.61818 + 0.1\left[3.1818 + 1 + \frac{3.2698}{1.4}\right]$

$= 2.61818 + 0.1\left[3.1818 + 1 + 2.335571429\right]$

$= \underline{3.2699}$

Thus $\underline{y(1.4) = 3.2699}$

Now, let us find the analytical solution of the equation

$\frac{dy}{dx} = 1 + \frac{y}{x}$ or $\frac{dy}{dx} - \frac{y}{x} = 1$

This is a linear DE of the form $\frac{dy}{dx} + Py = Q$

whose solution is given by

$y e^{\int P\,dx} = \int Q e^{\int P\,dx} dx + C$.

here $P = -1/x$ & $Q = 1$

$e^{\int P.dx} = e^{\int -1/x\,dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$

Solution becomes

$y \cdot 1/x = \int 1 \cdot 1/x\,dx + C = \int 1/x\,dx + C$

$y/x = \log x + C$ — (*)

apply the initial condition

i.e $y = 2$ & $x = 1$ we have

$2/1 = \log(1) + C \Rightarrow 2 = 0 + C \Rightarrow C = 2$

put c value in (*)

$$y/x = \log x + 2$$

$$y = x(\log x + 2)$$

This is the analytical solution of given initial value problem

Now by putting $x = 1.2$ & 1.4 in the above we get $y = 1.2(\log_e(1.2) + 2)$

$$y = 2.61878$$

put $x = 1.4$

$$y = 1.4(\log_e(1.4) + 2)$$

$$y = 3.27106$$

② ⓐ Using modified Euler's method find y at $x = 0.2$ given $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $y(0) = 1$

Dec 2017 taking $h = 0.1$. Perform three iterations at each step.

Sol^n: We need to find $y(0.2)$ by taking $h = 0.1$
The Problem has to be done in two stages

I Stage: By data $x_0 = 0$, $y_0 = 1$ $h = 0.1$

$$\frac{dy}{dx} = f(x, y) = 3x + \frac{1}{2}y$$

$$f(x_0, y_0) = 3(0) + \frac{1}{2}(1)$$

$$f(x_0, y_0) = \frac{1}{2} = 0.5$$

$$x_1 = x_0 + h = 0 + 0.1$$

$$x_1 = 0.1$$

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$$y(x_1) = y_1 = y(0.1) = ?$$

from Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$= 1 + (0.1)(0.5)$$

$$y_1^{(0)} = \underline{1.05}$$

we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{1.05}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 0.3 + 0.525\right]$$

$$y_1^{(1)} = 1.06625$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(1)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{1.06625}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 0.3 + 0.533125\right]$$

$$= 1.06665$$

$$y\ 1.0667$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.05\left[0.5 + 3x_1 + \frac{y_1^{(1)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{1.0667}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 0.3 + 0.53335\right]$$

$$= 1.06666$$

$$= 1.0667$$

Thus $y(0.1) = 1.0667$

II Stage: Now, let $x_0 = 0.1$, $y_0 = 1.0667$

we have $f(x, y) = 3x + \frac{y}{2}$

$$f(x_0, y_0) = 3(0.1) + \frac{1.0667}{2} = 0.83335$$

$x_1 = x_0 + h = 0.2$; $y_1 = y(x_1) = y(0.2) = ?$

From Euler's formula we obtain

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 1.0667 + 0.1(0.83335)$$

$$= 1.150035$$

$$\simeq 1.15$$

From modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1.0667 + \frac{0.1}{2}\left[0.83335 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{1.15}{2}\right]$$

$$y_1^{(1)} = 1.1671$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3x_1 + \frac{y_1^{(1)}}{2}\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{1.1671}{2}\right]$$

$$y_1^{(2)} = 1.1675$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3x_1 + \frac{y_1^{(2)}}{2}\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{1.1675}{2}\right]$$

$$y_1^{(3)} = 1.1675$$

Thus $y(0.2) = 1.1675$

③ Using modified Euler's method find $y(0.2)$ correct to four decimal places solving the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ taking $h = 0.1$

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(4) Using modified Euler's method find $y(20.2)$ and $y(20.4)$ given that $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$ with $y(20) = 5$ taking $h = 0.2$

Soln: we shall first find $y(20.2)$ and use this value to find $y(20.4)$

I Stage: By data

$x_0 = 20, \ y_0 = 5$ and $h = 0.2$

$f(x,y) = \log_{10}\left(\frac{x}{y}\right)$; $f(x_0, y_0) = \log_{10}(20/5)$

$f(x_0, y_0) = \log_{10}(4) = 0.60205$

$f(x_0, y_0) = 0.6021$

$x_1 = x_0 + h = 20 + 0.2 = 20.2$

$y(x_1) = y_1 = y(20.2) = ?$

From Euler's formula : $y_1^{(0)} = y_0 + hf(x_0, y_0)$

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$
$$= 5 + 0.2(0.6021)$$
$$= 5.1204$$

By Euler's modified formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 5 + \frac{0.2}{2}\left[0.6021 + \log_{10}\left(x_1/y_1^{(0)}\right)\right]$$

$$= 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1204}\right)\right]$$

$$y_1^{(1)} = 5.1198$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{x_1}{y_1^{(1)}}\right)\right]$$

$$= 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1198}\right)\right]$$

$$= \underline{5.1198}$$

Thus $y(20.2) = \underline{5.1198}$

II Stage: Let $x_0 = 20.2$ $y_0 = 5.1198$

$$f(x, y) = \log_{10}(x/y)$$

$$f(x_0, y_0) = \log_{10}\left(\frac{20.2}{5.1198}\right)$$

$$f(x_0, y_0) = 0.59609$$

$x_1 = x_0 + h = 20.4$, $y(x_1) = y_1 = y(20.4) = ?$

From Euler formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 5.1198 + 0.2(0.59609)$$

$$= \underline{5.2390}$$

$$y_1^{(0)} \approx \underline{5.239}$$

By modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 5.1198 + 0.1\left[0.59609 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$= 5.1198 + 0.1\left[0.59609 + \log_{10}\left(\frac{20.4}{5.239}\right)\right]$$

$y_1^{(1)} = 5.2384$

$y_1^{(2)} = y_0 + h/_2 \left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$= 5.1198 + 0.1 \left[0.59609 + \log_{10}\left(x_1/y_1^{(1)}\right)\right]$

$= 5.1198 + 0.1 \left[0.59609 + \log_{10}\left(\frac{20.4}{5.2384}\right)\right]$

$= 5.2384$

Thus $y(20.4) = 5.2384$

(5) Use modified Euler's method to solve $\frac{dy}{dx} = x + |\sqrt{y}|$ in the range $0 \le x \le 0.4$

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by taking $h = 0.2$ given that $y = 1$ at $x = 0$ initially.

Sol^n: we need to find $y(0.2)$ & $y(0.4)$ with $h = 0.2$

I stage: By data $x_0 = 0$ $y_0 = 1$ $f(x, y) = x + \sqrt{y}$, $h = 0.2$ where the modulus sign indicates that we have to take only the +ve value of $\sqrt{y}$.

$f(x_0, y_0) = x_0 + \sqrt{y_0} = 0 + 1 = 1$

$\therefore f(x_0, y_0) = 1$

$x_1 = x_0 + h = 0.2 \Rightarrow x_1 = 0.2$

$y(x_1) = y_1 = y(0.2) = ?$

From

Euler's formula

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1 + 0.2(1)$

$= 1.2$

By modified Euler's formula

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 1 + \frac{0.2}{2}\left[1 + x_1 + \sqrt{y_1^{(0)}}\right]$

$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.2}\right]$

$= \underline{1.2295}$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$= 1 + 0.1\left[1 + x_1 + \sqrt{y_1^{(1)}}\right]$

$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.2295}\right]$

$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.2295}\right]$

$= \underline{1.23088}$

$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$

$= 1 + 0.1\left[1 + x_1 + \sqrt{y_1^{(2)}}\right]$

$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.23088}\right]$

$= \underline{1.2309}$

$y_1^{(4)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$

$= 1 + 0.1\left[1 + 0.2 + \sqrt{1.2309}\right]$

$= \underline{1.2309}$

Thus $y(0.2) = \underline{1.2309}$

II stage: Now let $x_0 = 0.2$ $y_0 = 1.2309$

$f(x, y) = x + \sqrt{y}$

$f(x_0, y_0) = x_0 + \sqrt{y_0} = 0.2 + \sqrt{1.2309} = 1.30945$

$f(x_0, y_0) = 1.3095$

$x_1 = x_0 + h$

$x_1 = 0.2 + 0.2 \Rightarrow x_1 = 0.4$

$y(x_1) = y_1 = y(0.4) = ?$

from Eulers formula

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

$= 1.2309 + (0.2)(1.3095)$

$= \underline{\underline{1.4928}}$

from modified Euler's formula

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$

$= 1.2309 + \frac{0.2}{2}\left[1.3095 + x_1 + \sqrt{y_1^{(0)}} \right]$

$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.4928} \right]$

$= \underline{\underline{1.5240}}$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$

$= 1.2309 + 0.1\left[1.3095 + x_1 + \sqrt{y_1^{(1)}} \right]$

$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.5240} \right]$

$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.5240} \right]$

$= \underline{\underline{1.5253}}$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)}) \right]$

$= 1.2309 + 0.1\left[1.3095 + x_1 + \sqrt{y_1^{(2)}} \right]$

$= 1.2309 + 0.1\left[1.3095 + 0.4 + \sqrt{1.5253} \right]$

$= \underline{1.5253}$

thus $\underline{\underline{y(0.4) = 1.5253}}$

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Do yourself

⑥ Use modified Euler's method to compute $y(0.1)$ given that $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by taking $h = 0.05$ considering the accuracy upto two approximation in each step.

Soln: we need to compute $y(0.05)$ first and use this value to compute $y(0.1)$

I Stage: By data $x_0 = 0$, $y_0 = 1$

$f(x, y) = x^2 + y$, $h = 0.05$

$f(x_0, y_0) = x_0^2 + y_0 = 0^2 + 1$

$f(x_0, y_0) = 1$

$x_1 = x_0 + h = 0 + 0.05 \Rightarrow x_1 = 0.05$

$y(x_1) = y_1 = y(0.05) = ?$

From Euler's formula

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1 + 0.05(1)$

$= 1.05$

From modified Euler's formula

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 1 + \frac{0.05}{2}\left[1 + (x_1)^2 + y_1^{(0)}\right]$

$= 1 + 0.025\left[1 + (0.05)^2 + 1.05\right]$

$y_1^{(1)} = 1.0513$

$y_1^{(2)} = 1 + 0.025\left[1 + (0.05)^2 + y_1^{(1)}\right]$

$= 1 + 0.025\left[1 + (0.05)^2 + 1.0513\right]$

$= 1.0513$

Thus $y(0.05) = 1.0513$

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II stage: Now let $x_0 = 0.05$, $y_0 = 1.0513$

$f(x, y) = x^2 + y$;

$f(x_0, y_0) = x_0^2 + y_0 = (0.05)^2 + (1.0513)$

$f(x_0, y_0) = 1.0538$

$x_1 = x_0 + h = 0.05 + 0.05 = 0.1 \Rightarrow x_1 = 0.1$

$y(x_1) = y_1 = y(0.1) = ?$

Now, By Euler's formula

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1.0513 + 0.05(1.0538)$

$= 1.10399$

Now, From modified Euler's formula

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 1.0513 + \frac{0.05}{2}\left[1.0538 + x_1^2 + y_1^{(0)}\right]$

$= 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.10399\right]$

$= 1.10389$

$y_1^{(1)} = 1.10389$

$y_1^{(2)} = 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.10389\right]$

$= 1.10549$

$y_1^{(3)} = 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.10549\right]$

$= 1.1055$

$y_1^{(4)} = 1.0513 + 0.025\left[1.0538 + (0.1)^2 + 1.1055\right]$

$= 1.1055$

Thus $y(0.1) = 1.1055$

Do yourself

(7) Using Euler's predictor and corrector formula solve $\frac{dy}{dx} = x + y$ at $x = 0.2$ given that $y(0) = 1$

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Ans: $x_0 = 0,\ y_0 = 1 \quad f(x_0, y_0) = x_0 + y_0 = 1$

$x_1 = x_0 + h = 0.2$

$x_1 = x_0 + h \Rightarrow h = x_1 - x_0$

$h = 0.2 - 0$

$h = 0.2$

$y(x_1) = y_1 = y(0.2) = ?$

we have euler's formula

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1 + (0.2)(1)$

$= 1.2$

we have from modified Euler's formula

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 1 + \frac{0.2}{2}\left[1 + x_1 + y_1^{(0)}\right]$

$= 1 + 0.1\left[1 + 0.2 + 1.2\right]$

$= 1.24$

$y_1^{(2)} = 1.244, \quad y_1^{(3)} = 1.2444$

They $y(0.2) = 1.2444$

(8) Using Euler's predictor and corrector formula compute $y(1.1)$, correct to five decimal places given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y = 1$ at $x = 1$. Also find the analytical solⁿ.

Solⁿ: By data $x_0 = 1,\ y_0 = 1$

$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$

$\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$

$= \frac{1 - yx}{x^2}$

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we have $f(x, y) = \frac{1 - yx}{x^2}$

$$f(x_0, y_0) = \frac{1 - y_0 x_0}{x_0^2} = \frac{1 - (1)(1)}{1^2} = 0$$

$1+h = 1 \cdot 1 \Rightarrow h = 1 \cdot 1 - 1 \Rightarrow h = 0 \cdot 1$

$x_1 = x_0 + h = 1 + 0 \cdot 1 = 1 \cdot 1 \Rightarrow x_1 = 1 \cdot 1$

$y(x_1) = y_1 = y(1 \cdot 1) = ?$

From Euler's formula:

$y_1^{(0)} = y_0 + h f(x_0, y_0)$

$y_1^{(0)} = 1 + (0 \cdot 1)(0)$

$y_1^{(0)} = 1$ //

we have modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$

$$= 1 + \frac{0 \cdot 1}{2} \left[0 + \frac{1 - x_1 y_1^{(0)}}{x_1^2} \right]$$

$$= 1 + 0 \cdot 05 \left[\frac{1 - (1 \cdot 1)(1)}{(1 \cdot 1)^2} \right]$$

$y_1^{(1)} = 0 \cdot 99586$

$$y_1^{(2)} = y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$$

$$= 1 + \frac{0 \cdot 1}{2} \left[0 + \frac{1 - x_1 y_1^{(1)}}{x_1^2} \right]$$

$$= 1 + 0 \cdot 05 \left[\frac{1 - (1 \cdot 1)(0 \cdot 99586)}{(1 \cdot 1)^2} \right]$$

$= \underline{0 \cdot 99605}$

$$y_1^{(3)} = y_0 + \frac{h}{2} \left[f(x_0, y_0) + f(x_1, y_1^{(2)}) \right]$$

$$= 1 + 0 \cdot 05 \left[0 + \frac{1 - x_1 y_1^{(2)}}{x_1^2} \right]$$

$$= 1 + 0.05 \left[\frac{1 - (1.1)(0.99605)}{(1.1)^2} \right]$$

$$= 0.9960421$$

$$= 0.99605$$

Thus $y(1.1) = 0.99605$

Analytical solution

$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ is of the form $\frac{dy}{dx} + Py = Q$

$P = \frac{1}{x}$ and $Q = \frac{1}{x^2}$

whose solution is given by

$$y(IF) = \int Q \, IF \, dx + C$$

where $IF = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$\therefore IF = x$

Sol^n is $y(IF) = \int Q \, IF \, dx + C$

$$y(x) = \int \frac{1}{\cancel{x^2}} \times \cancel{x} \, dx + C$$

$$xy = \int \frac{1}{x} dx + C$$

$$xy = \log x + C \quad — (*)$$

Now we have to find $y(1.1)$

by using the initial condition

$x = 1 \quad y = 1$

$(1)(1) = \log(1) + C \Rightarrow 1 = 0 + C \Rightarrow C = 1$

$[\log(1) = 0]$

put c value in (*)

$xy = \log x + 1$

or $y = \frac{1+\log x}{x}$

put $x = 1.1$ we will get $y(1.1)$

$y = \frac{1+\log(1.1)}{1.1}$

$y = \underline{0.995736}$ [use ln]

$y = \underline{0.99574}$ is the analytical solution

Runge Kutta method of fourth order

Consider the initial value problem

$\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

we need to find $y(x_0+h)$ where h is the step size.

we have to first compute K_1, K_2, K_3, K_4 by the following formula

$K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$

$K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$

$K_4 = hf(x_0+h, y_0+K_3)$

The required

$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Problems

1) Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, compute $y(0.2)$ by taking $h = 0.2$ using Range-Kutta method of fourth order. Also find the analytical solution.

Sol^n: By data $f(x, y) = \frac{dy}{dx} = 3x + \frac{y}{2}$

$\therefore f(x, y) = 3x + \frac{y}{2}$,

$x_0 = 0 \quad y_0 = 1 \quad h = 0.2 \quad f(x_0, y_0) = 0.5$

we shall first find K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.2) f(0, 1)$

$K_1 = (0.2)(0.5)$

$K_1 = 0.1$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$

$= 0.2 f(0.1, 1.05)$

$= 0.2 \left[3(0.1) + \frac{1.05}{2}\right]$

$= 0.165$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$

$= 0.2 f(0.1, 1.0825)$

$= 0.2 \left[3(0.1) + \frac{1.0825}{2}\right]$

$K_3 = 0.16825$ //

$K_4 = hf(x_0+h, y_0+K_3)$
$= 0.2f(0+0.2, 1+0.16825)$

$= 0.2f(0.2, 1.16825)$

$= 0.2\left[3(0.2) + \frac{1.16825}{2}\right]$

$K_4 = 0.236825$

$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$y(0.2) = 1 + \frac{1}{6}(0.1 + 2(0.165) + 2(0.16825) + 0.236825)$

$= 1.1672$

we shall find the analytical solution of the given equation by writing in the form $\frac{dy}{dx} + Py = Q$ whose solution is

$y(IF) = \int Q \cdot IF \, dx + C$

where $IF = e^{\int P \cdot dx}$ where $\frac{dy}{dx} - \frac{y}{2} = 3x$

here $P = -\frac{1}{2}$ $Q = 3x$

$IF = e^{\int P \cdot dx} = e^{\int -\frac{1}{2} dx}$

$IF = e^{-\frac{1}{2}x}$

Soln is $ye^{-\frac{1}{2}x} = \int 3x e^{-x/2} dx + C$

$ye^{-x/2} = 3\int x e^{-x/2} dx + C$

Integrating RHS by parts we have

$$ye^{-x/2} = 3\left[x\frac{e^{-x/2}}{-1/2} - \int \frac{e^{-x/2}}{-1/2}(1)\,dx\right] + c$$

$$= 3\left[-2xe^{-x/2} + 2\frac{e^{-x/2}}{-1/2}\right] + c$$

$$ye^{-x/2} = -6xe^{-x/2} - 12e^{-x/2} + c$$

$\div$ B.S by $e^{-x/2}$

$$y = -6x\cancel{\frac{e^{-x/2}}{e^{-x/2}}} - 12\cancel{\frac{e^{-x/2}}{e^{-x/2}}} + \frac{c}{e^{-x/2}}$$

$$y = -6x\cancel{e^{-x/2}}\cdot\cancel{e^{x/2}} - 12 + ce^{x/2}$$

$$\therefore y = -6x - 12 + ce^{x/2}$$

We initial condition to find c, $x=0, y=1$, $1 = 0 - 12 + ce^{0}$

$\Rightarrow$ $c = 13$, Now by putting $x = 0.2$ we have

$$y(0.2) = -6(0.2) - 12 + 13e^{0.2/2}$$

$$y(0.2) = 1.1672219$$

$y(0.2) = 1.1672$ by analytical solution

(2) Do yourself

Use fourth order Runge Kutta method to solve $(x+y)\frac{dy}{dx} = 1$, $y(0.4) = 1$ at $x = 0.5$ Correct to four decimal places

Solⁿ: $\frac{dy}{dx} = \frac{1}{x+y}$, $y_0 = 1$, $x_0 = 0.4$, $y(0.5) = ?$

$x = x_0 + h$

$x = 0.4 + h$

$0.5 - 0.4 = h \Rightarrow h = 0.1$

$K_1 = 0.0714$

$K_2 = 0.0673$

$K_3 = 0.0674$

$K_4 = 0.0638$

$y(0.5) = 1.0674$ //

Dec 17 June 18 ③ Using Runge Kutta method of fourth order find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$

$y(0) = 1$ taking $h = 0.2$

Solⁿ: By data

$f(x, y) = \frac{y-x}{y+x}$ $x_0 = 0$ $y_0 = 1$ $h = 0.2$

$y(0.2) = ?$

we shall find K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.2) f(0, 1)$

$= (0.2)\left[\frac{1-0}{1+0}\right]$

$K_1 = 0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1 + 0.1)$

$= (0.2) f(0.1, 1.1)$

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$$K_2 = (0.2)\left[\frac{1.1-0.1}{1.1+0.1}\right]$$

$$K_2 = 0.1667$$

$$K_3 = hf\left(x_0+h/2,\ y_0+\frac{K_2}{2}\right)$$

$$= 0.2f\left(0+\frac{0.2}{2},\ 1+\frac{0.1667}{2}\right)$$

$$= 0.2f(0.1,\ 1.08335)$$

$$= 0.2\left[\frac{1.08335-0.1}{1.08335+0.1}\right]$$

$$= 0.2\left[\frac{0.98335}{1.18335}\right]$$

$$= 0.16619$$

$$K_4 = hf(x_0+h,\ y_0+K_3)$$

$$= 0.2f(0+0.2,\ 1+0.16619)$$

$$= 0.2f(0.2,\ 1.16619)$$

$$= 0.2\left[\frac{1.16619-0.2}{1.16619+0.2}\right]$$

$$= 0.1414$$

$$y(x_0+h) = y_0 + 1/6\,(K_1+2K_2+2K_3+K_4)$$

$$y(0.2) = 1 + 1/6\,(0.2+2(0.1667)+2(0.16619)+0.1414)$$

$y(0.2) = 1.16786$

(ii) Use fourth order Runge Kutta method to find y at $x = 0.1$ given that $\frac{dy}{dx} = 3e^{x} + 2y$, $y(0) = 0$ and $h = 0.1$

Soln: By data $f(x, y) = 3e^{x} + 2y$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$

$y(0.1) = ?$

$K_1 = hf(x_0, y_0) = 0.1 f(0, 0) = (0.1)(3e^{0} + 2(0))$

$K_1 = (0.1)(3)$

$K_1 = 0.3$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$= 0.1 f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3}{2}\right)$

$= 0.1 f(0.05, 0.15)$

$= 0.1[3e^{0.05} + 2(0.15)]$

$= 0.34538$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$= 0.1 f\left(0.05, 0 + \frac{0.34538}{2}\right)$

$= 0.1 f(0.05, 0.17269)$

$= 0.1[3e^{0.05} + 2(0.17269)]$

$= 0.3499$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1 f(0.1, 0 + 0.3499)$

~~$= 0.1 f(0.05, 0.17449)$~~

$= 0.1 \left[3e^{0.95} + 2(0.17495)\right]$

$= 0.1 \left[3e^{0.01} + 2(0.3499)\right]$

$= 0.4015$

$y(x_0+h) = y_0 + 1/6\,(K_1 + 2K_2 + 2K_3 + K_4)$

$= 0 + 1/6\,(0.3 + 2(0.3454) + 2(0.3499) + 0.4015)$

$y(0.1) = 0.34868$

Do yourself

(5) Use fourth order Runge Kutta method to find $y(1.1)$ given that $\frac{dy}{dx} = xy^{1/3}$, $y(1)=1$

Sol^n: By data $f(x,y) = xy^{1/3}$ $x_0 = 1, y_0 = 1$

we need to find $y(1.1)$

$x_0 + h = 1.1$

$h = 1.1 - x_0$

$h = 1.1 - 1$

$h = 0.1$

we shall find K_1, K_2, K_3, K_4

$K_1 = 0.1$

$K_2 = 0.1067$ $\qquad y(1.1) = 1.1068$

$K_3 = 0.1068$

$K_4 = 0.1138$

Do yourself

(6) Using Runge Kutta method of fourth order solve $\frac{dy}{dx} + y = 2x$ at $x = 1.1$ given that $y=3$ at $x=1$ initially

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By data

Sol^n: $\frac{dy}{dx} = 2x - y,\ y_0 = 3\ \ x_0 = 1$

$f(x,y) = 2x - y,\ x_0 + h = 1.1 \Rightarrow h = 1.1 - x_0 = 1.1 - 1$

$h = 0.1$//

we shall find K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.1) f(1,3) = 0.1[2(1) - 3]$

$K_1 = -0.1$

$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$

$= (0.1) f\left(1 + \frac{0.1}{2},\ 3 + \frac{(-0.1)}{2}\right)$

$= (0.1) f(1.05,\ 2.95)$

$= (0.1)(2(1.05) - 2.95)$

$= -0.085$//

$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$

$= 0.1 f\left(1.05,\ 3 + \frac{(-0.085)}{2}\right)$

$= 0.1 f(1.05,\ 2.9575)$

$= 0.1[2(1.05) - 2.9575]$

$= -0.08575$//

$K_4 = hf(x_0 + h,\ y_0 + K_3)$

$= 0.1 f(1 + 0.1,\ 3 + (-0.08575))$

$= 0.1 f(1.1, 3 + (-0.08575))$

$= 0.1 f(1.1,\ 2.91425)$

$= 0.1[2(1.1) + 2.91425]$

$K_4 = \underline{\underline{-0.071425}}$

$$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 3 + \frac{1}{6}(-0.1 + 2(-0.085) + 2(-0.08575) - 0.071425)$$

$$y(1.1) = 2.9145125 \simeq \underline{\underline{2.9145}}$$

(7) Using Runge Kutta method of fourth order find $y(0.2)$ for the equation

DEC 2018

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0)=1 \quad \text{taking } h=0.1$$

Sol^n: The problem has to be done in two stages

I stage: $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

we have to find $y(0.1)$

we shall find K_1, K_2, K_3, K_4

$$K_1 = hf(x_0, y_0)$$

$$= (0.1) f(0,1)$$

$$= (0.1)\left[\frac{1-0}{1+0}\right]$$

$$= (0.1)$$

$$K_1 = 0.1$$

$$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$$

$$= (0.1) f(0+0.05,\ 1+0.05)$$

$$= (0.1) f(0.05,\ 1.05)$$

$$= (0.1)\left[\frac{1.05-0.05}{1.05+0.05}\right]$$

$$= \underline{\underline{0.0909}}$$

$$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$$
$$= (0.1)\, f\left(0 + 0.05,\ 1 + \frac{0.0909}{2}\right)$$
$$= (0.1)\, f(0.05,\ 1.04545)$$
$$= (0.1)\left[\frac{1.04545 - 0.05}{1.04545 + 0.05}\right]$$
$$= 0.09087$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3)$$
$$= (0.1)\, f(0 + 0.1,\ 1 + 0.09087)$$
$$= (0.1)\, f(0.1,\ 1.09087)$$
$$= 0.0832$$

$$y(x_0 + h) = y_0 + 1/6\,(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= 1 + 1/6\,(0.1 + 2(0.0909) + 2(0.09087) + 0.0832)$$
$$y(0.1) = 1.09112$$

II Stage: $f(x, y) = \frac{y - x}{y + x}$, $x_0 = 0.1$ $y_0 = 1.09112$ $h = 0.1$

we need to find $y(0.2)$

$$K_1 = hf(x_0, y_0)$$
$$= (0.1)\, f(0.1,\ 1.09112)$$
$$= 0.1\left[\frac{1.09112 - 0.1}{1.09112 + 0.1}\right]$$
$$= 0.0832$$

$$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$$
$$= (0.1)\, f\left(0.1 + \frac{0.1}{2},\ 1.09112 + \frac{0.0832}{2}\right)$$
$$= (0.1)\, f(0.15,\ 1.13272)$$
$$= (0.1)\left[\frac{1.13272 - 0.15}{1.13272 + 0.15}\right]$$
$$= 0.0766$$

$$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$$
$$= (0.1)\, f\left(0.1 + \frac{0.1}{2},\ 1.09112 + \frac{0.0766}{2}\right)$$
$$= (0.1)\, f(0.15,\ 1.12942)$$

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$= (0.1) f(0.15, \ 1.12942)$

$= (0.1) \left[\dfrac{1.12942 - 0.15}{1.12942 + 0.15} \right]$

$= \underline{\underline{0.07655}}$

$K_4 = hf(x_0 + h, \ y_0 + K_3)$

$= (0.1) f(0.1 + 0.1, \ 1.09112 + 0.07655)$

$= (0.1) f(0.2, \ 1.16767)$

$= (0.1) \left[\dfrac{1.16767 - 0.2}{1.16767 + 0.2} \right]$

$K_4 = \underline{\underline{0.07075}}$

$y(x_0 + h) = y_0 + \dfrac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$

$= 1.09112 + \dfrac{1}{6} (0.0832 + 2(0.0766) + 2(0.07655) + 0.07075)$

$y(0.2) = \underline{\underline{1.167828}}$

(8) Solve: $(y^2 - x^2) dx = (y^2 + x^2) dy$ for $x = 0.2$ and 0.4 given that $y = 1$ at $x = 0$ initially, by applying Runge Kutta method of order 4.

June 2017

Soln: we have $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ $\quad x_0 = 0, \ y_0 = 1 \quad h = 0.2$

here we have to find $y(0.2)$

I Stage: $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$ we shall find K_1, K_2, K_3, K_4

$K_1 = hf(x_0, y_0) = (0.2) f(0, 1)$

$= (0.2) \left[\dfrac{1^2 - 0^2}{1^2 + 0^2} \right]$

$K_1 = (0.2)$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1.1)$

$= (0.2) \left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right]$

$= \underline{\underline{0.19672}}$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.19672}{2}\right)$

$= (0.2) f(0.1, 1.09836)$

$= (0.2) \left[\frac{(1.09836)^2 - (0.1)^2}{(1.09836)^2 + (0.1)^2}\right]$

$= \underline{\underline{0.19671}}$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.2) f(0 + 0.2, 1 + 0.19671)$

$= (0.2) f(0.2, 1.19671)$

$= (0.2) \left[\frac{(1.19671)^2 - (0.2)^2}{(1.19671)^2 + (0.2)^2}\right]$

$= 0.18913$

we have $y(x_0 + h) = y_0 + 1/6 (K_1 + 2K_2 + 2K_3 + K_4)$

$= 1 + \frac{1}{6}(0.2 + 2(0.19672) + 2(0.19671) + 0.18913)$

$y(0.2) = \underline{\underline{1.1959}}$

<u>II Stage</u>: Now we have to find $y(0.4)$

$f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$ $x_0 = 0.2$, $y_0 = 1.1959$, $h = 0.2$

$K_1 = hf(x_0 + y_0)$

$= (0.2) f(0.2, 1.1959)$

$= 0.2 f(0.2, 1.1959)$

$$= (0.2)\left[\frac{(1.1959)^2 - (0.2)^2}{(1.1959)^2 + (0.2)^2}\right]$$

$= 0.1891$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$$= (0.2) f\left(0.2 + \frac{0.2}{2}, 1.1959 + \frac{0.1891}{2}\right)$$

$= (0.2) f(0.3, 1.29045)$

$$= (0.2)\left[\frac{(1.29045)^2 - (0.3)^2}{(1.29045)^2 + (0.3)^2}\right]$$

$= 0.17949$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$$= (0.2) f\left(0.2 + \frac{0.2}{2}, 1.1959 + \frac{0.17949}{2}\right)$$

$= (0.2) f(0.3, 1.285645)$

$$= (0.2)\left[\frac{(1.285645)^2 - (0.3)^2}{(1.285645)^2 + (0.3)^2}\right]$$

$= 0.17934$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.2) f(0.2 + 0.2, 1.1959 + 0.17934)$

$= (0.2) f(0.4, 1.37524)$

$$= (0.2)\left[\frac{(1.37524)^2 - (0.4)^2}{(1.37524)^2 + (0.4)^2}\right]$$

$= 0.1688$

$$y(x_0+h) = y_0 + \tfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= 1.1959 + \tfrac{1}{6}(0.1891 + 2(0.17949) + 2(0.17934) + 0.1688)$$
$$y(0.4) = 1.37516$$

Numerical Predictor and Corrector Methods

We discuss two predictor and corrector methods namely
① Milne's method ② Adams - Bashforth method
Consider the differential equation $y' = \frac{dy}{dx} = f(x,y)$ with a set of four predetermined values of y; $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$ and $y(x_3) = y_3$
here x_0, x_1, x_2, x_3 are equally spaced values of x with width h

Predictor and corrector formula to compute $y(x_4) = y_4$ are as follows

Milne's predictor and Corrector formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \quad \text{... (predictor formula)}$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \quad \text{... (corrector formula)}$$

Adams - Bashforth Predictor and corrector formula

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0') \quad \text{(predictor formula)}$$

$$y_4^{(c)} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1') \quad \text{(corrector formula)}$$

Working procedure

1. We first prepare the table showing the values of y corresponding to four equidistant values of x and the computation of $y' = f(x, y)$
2. We compute y_4 from the predictor formula
3. We use this value of y_4 to compute $y_4' = f(x_4, y_4)$
4. We apply corrector formula to obtain the corrected value of y_4
5. This value is used for computing y_4' to apply the corrector formula again
6. The process is continued till we get consistency in two consecutive values of y_4

NOTE: We can also find $y_5, y_6 \ldots$ by deducing expressions from the general form of predictor and corrector formula.

Problems

1. Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0) = 0$,

June 2018 $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$

Compute y at $x = 0.8$ by applying

1. Milne's method
2. Adams-Bashforth method

Soln: We prepare the following table using the given data which is essentially required for applying the predictor and corrector formula

P.T.O

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y'_0 = 0 - 0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

(a) By milne's method

we have the predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$= 0 + \frac{4(0.2)}{3}(2(0.1996) - 0.3937 + 2(0.5689))$$

$$= 0.30488$$

$$y'_4 = x_4 - y_4^2$$

$$= 0.8 - (0.30488)^2$$

$$= 0.707$$

we have the corrector formula

$$y_4^{(C)} = y_2 + h/3\,(y'_2 + 4y'_3 + y'_4)$$

$$= 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.707)$$

$$= 0.30458$$

Now, $y'_4 = x_4 - y_4^2$

$$= 0.8 - (0.30458)^2$$

$$= 0.7072$$

Substituting the value of y'_4 again in the Corrector formula

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7072]$$

$$y_4^{(C)} = 0.3046$$

$\therefore \; y_4 = y(0.8) = 0.3046$

ⓑ By Adams - Bashforth method

we have predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0')$$

$$= 0.1762 + \frac{0.2}{24}\left[55(0.5689) - 59(0.3937) + 37(0.1996) - 9(0)\right]$$

$$y_4^{(P)} = 0.30492$$

Now, $y_4' = x_4 - y_4^2$

$$y_4' = 0.8 - (0.30492)^2$$

$$y_4' = 0.707$$

Next, we have the corrector formula

$$y_4^{(c)} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$y_4^{(c)} = 0.1762 + \frac{0.2}{24}\left[9(0.707) + 19(0.5689) - 5(0.3937) + 0.1996\right]$$

$$y_4^{(c)} = 0.30456$$

$$y_4' = x_4 - y_4^2$$

$$= 0.8 - (0.30456)^2$$

$$\simeq 0.7072$$

Applying corrector formula again with only change in the value of y_4' we obtain,

$$y_4^{(c)} = 0.1762 + \frac{0.2}{24}\left[9(0.7072) + 19(0.5689) - 5(0.3937) + 0.1996\right]$$

$y_4^{(1)} = 0.30456$

Thus $y_4 = y(0.8) = 0.30456$

(8) Apply milne's method to compute $y(1.4)$ correct to four decimal places and following data given $\frac{dy}{dx} = x^2 + y/2$ and $y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4649$, $y(1.3) = 2.7514$

Solⁿ: First we shall prepare the following table

x	y	$y' = x^2 + y/2$
$x_0 = 1$	$y_0 = 2$	$y_0' = 1^2 + 2/2 = 1+1 = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = (1.1)^2 + \frac{2.2156}{2} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y_2' = (1.2)^2 + \frac{2.4649}{2} = 2.67245$
$x_3 = 1.3$	$y_3 = 2.7514$	$y_3' = (1.3)^2 + \frac{2.7514}{2} = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	

we have $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$$y_4^{(P)} = 2 + \frac{4(0.1)}{3}\left[2(2.3178) - 2.67245 + 2(3.0657)\right]$$

$= 3.07927$

$\therefore y_4' = x_4^2 + \frac{y_4}{2}$

$= (1.4)^2 + \frac{3.07927}{2}$

$y_4' = 3.49963$

Now consider

$$y_4^{(c)} = y_2 + h/3\,(y_2' + 4y_3' + y_4')$$

$$= 2.4649 + \frac{0.1}{3}\left[2.67245 + 4(3.0657) + 3.49963\right]$$

$$y_4^{(c)} = 3.07939$$

Now, $y_4' = x_4^2 + y_4/2$

$$= (1.4)^2 + \frac{3.07939}{2}$$

$$= 3.49969$$

Substituting this value of y_4' again in the corrector formula we obtain

$$y_4^{(c)} = 2.4649 + \frac{0.1}{3}\left[2.67245 + 4(3.0657) + 3.49969\right]$$

$$y_4^{(c)} = 3.07939$$

Thus $y_4 = y(1.4) = 3.07939$

(3) If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ and $y(0.3) = 2.090$ find $y(0.4)$ correct to four decimal places by using

(a) Milne's predictor - corrector method

(b) Adams - Bashforth predictor - corrector method

(Apply the corrector formula twice)

x	y	$y' = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 2e^0 - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 2e^{x_1} - y_1 = 2e^{0.1} - 2.010 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 2e^{x_2} - y_2 = 2e^{0.2} - 2.040 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 2e^{x_3} - y_3 = 2e^{0.3} - 2.090 = 0.6097$
$x_4 = 0.4$	$y_4 = ?$	

(a) By milne's predictor - corrector method

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 2 + \frac{4(0.1)}{3}(2(0.2003) - (0.4028) + 2(0.6097))$$

$$= 2.16229$$

$$y_4^{(P)} = 2.1623$$

Now, $y_4' = 2e^{x_4} - y_4$

$$= 2e^{0.4} - 2.1623$$

$$= 0.8213$$

Next, we have milne's corrector formula

$$y_4^{(c)} = y_2 + h/3(y_2' + 4y_3' + y_4')$$

$$= 2.040 + \frac{0.1}{3}(0.4028 + 4(0.6097) + 0.8213)$$

$$= 2.16209$$

$$y_4^{(c)} = 2.1621$$

$$y_4' = 2e^{x_4} - y_4$$

$$= 2e^{0.4} - 2.1621$$

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$y'_4 = 0.8215$

Applying corrector formula again we have

$$y_4^{(c)} = 2.04 + \frac{0.1}{3}\left[0.4028 + 4(0.6097) + 0.8215\right]$$

$$= 2.1621$$

Thus $y(0.4) = 2.1621$

(b) By Adams-Bashforth predictor-corrector method

we have $y_4^{(P)} = y_3 + h/_{24}\,(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0)$

$$y_4^{(P)} = 2.090 + \frac{0.1}{24}\left[55(0.6097) - 59(0.4028) + 37(0.2003) - 9(0)\right]$$

$$y_4^{(P)} = 2.16158$$

$$y_4^{(P)} = 2.1616$$

Now, $y'_4 = 2e^{x_4} - y_4$

$$y'_4 = 2e^{0.4} - 2.1616$$

$$y'_4 = 0.822$$

Next we have, $y_4^{(c)} = y_3 + h/_{24}\,(9y'_4 + 19y'_3 - 5y'_2 + y'_1)$

$$y_4^{(c)} = 2.090 + \frac{0.1}{24}\left[9(0.822) + 19(0.6097) - 5(0.4028) + 0.2003\right]$$

$$y_4^{(c)} = 2.1615$$

Now, $y'_4 = 2e^{0.4} - y_4$

$$y'_4 = 2e^{0.4} - 2.1615$$

$$= 0.822149 \Rightarrow y'_4 = 0.82215$$

Substituting again in the corrector formula, we obtain $y_4^{(1)} = 2.090 + \frac{0.1}{24}\left[9(0.88215) + 19(0.6097) - 5(0.4028) + 0.2003\right]$

$$y_4^{(1)} = 2.1615$$

Thus $y(0.4) = 2.1615$

(4) Apply Adams - Bashforth method to solve the equation $(y^2+1)\,dy - x^2\,dx = 0$ at $x = 1$ given $y(0) = 1$, $y(0.25) = 1.0026$, $y(0.5) = 1.0206$, $y(0.75) = 1.0679$. Apply corrector formula twice.

Soln: By data $\frac{dy}{dx} = y' = \frac{x^2}{y^2+1}$

we prepare the following table

x	y	$y' = \frac{x^2}{y^2+1}$
$x_0 = 0$	$y_0 = 1$	$y_0' = \frac{x_0^2}{y_0^2+1} = \frac{0}{1+1} = 0$
$x_1 = 0.25$	$y_1 = 1.0026$	$y_1' = \frac{x_1^2}{y_1^2+1} = \frac{(0.25)^2}{(1.0026)^2+1} = 0.03116$
$x_2 = 0.5$	$y_2 = 1.0206$	$y_2' = \frac{x_2^2}{y_2^2+1} = \frac{(0.5)^2}{(1.0206)^2+1} = 0.12245$
$x_3 = 0.75$	$y_3 = 1.0679$	$y_3' = \frac{x_3^2}{y_3^2+1} = \frac{(0.75)^2}{(1.0679)^2+1} = 0.2628$
$x_4 = 1$	$y_4 = ?$	

we have the predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_4^{(P)} = 1.0679 + \frac{0.25}{24}\left[55(0.2628) - 59(0.12245) + 37(0.03116) - 9(0)\right]$$

$$= 1.1552$$

$$y_4' = \frac{x_4^2}{y_4^2 + 1}$$

$$y_4' = \frac{1^2}{(1.1552)^2 + 1}$$

$$y_4' = 0.42835$$

$$y_4' = 0.4284$$

Next, we have the corrector formula

$$y_4^{(c)} = y_3 + \frac{h}{24}\left(9y_4' + 19y_3' - 5y_2' + y_1'\right)$$

$$= 1.0679 + \frac{0.25}{24}\left[9(0.4284) + 19(0.2628) - 5(0.12245) + 0\right]$$

$$= 1.1536$$

$$y_4^{(c)} = 1.154$$

Now, $$y_4' = \frac{1^2}{(1.154)^2 + 1}$$

$$y_4' = 0.4288$$

Applying the corrector formula again we obtain

$$y_4^{(c)} = 1.0679 + \frac{0.25}{24}\left[9(0.4288) + 19(0.2628) - 5(0.12245) + 0\right]$$

$$y_4^{(c)} = 1.1537 \Rightarrow y_4^{(c)} = 1.154$$

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Thus $y(1) = 1.154$

(6) The following table gives the solution of $5xy' + y^2 - 2 = 0$. Find the value of y at $x = 4.5$ using Milne's predictor and corrector formula. Use the corrector formula twice.

x	4	4.1	4.2	4.3	4.4	//////
y	1	1.0049	1.0097	1.0143	1.0187	//////

Soln: By data

$$5xy' + y^2 - 2 = 0$$

$$5xy' = 2 - y^2$$

$$y' = \frac{2 - y^2}{5x}$$

x	y	$y' = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \frac{2 - y_0^2}{5x_0} = \frac{2 - (1)^2}{5(4)} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = 0.047829$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5(4.2)} = 0.04669$
$x_3 = 4.3$	$y_3 = 1.0143$	$y_3' = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5(4.3)} = 0.04415$ 0.04517
$x_4 = 4.4$	$y_4 = 1.0187$	$y_4' = \frac{2 - y_4^2}{5x_4} = \frac{2 - (1.0187)^2}{5(4.4)} = 0.04373$
$x_5 = 4.5$	$y_5 = ?$	

We have Milne's predictor and corrector formula in the standard form

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right);$$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

Since we require y_5, the equivalent form of these formula are given by

$$y_5^{(P)} = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4'\right]$$

$$y_5^{(c)} = y_3 + h/3\left[y_3' + 4y_4' + y_5'\right]$$

hence $y_5^{(P)} = 1.0049 + \frac{4(0.1)}{3}\left[2(0.04669) - 0.04415 + 2\times 0.04373\right]$

$$= \underline{1.02312}$$

$$y_5' = \frac{2 - y_5^2}{5x_5}$$

$$= \frac{2 - (1.02312)^2}{5(4.5)}$$

$$= \underline{0.04236}$$

hence, $y_5^{(c)} = y_3 + h/3\left[y_3' + 4y_4' + y_5'\right]$

$$= 1.0143 + \frac{0.1}{3}\left[\cancel{0.04415}^{0.04517} + 4(0.04373) + 0.04236\right]$$

$$= 1.023048$$

$$\approx \underline{1.023}$$

$$y_5' = \frac{2 - y_5^2}{5x_5}$$

$$= \frac{2 - (1.023)^2}{5(4.5)}$$

$$y_5' = \underline{0.04237}$$

applying corrector formula again

$$y_5^{(1)} = y_3 + \frac{h}{3}\left[y'_3 + 4y'_4 + y'_5\right]$$

$$= 1.0143 + \frac{0.1}{3}\left[\cancel{0.04415}\ 0.04517 + 4(0.04373) + 0.04237\right]$$

$$= 1.023048$$

$$\approx 1.023$$

Thus $y(4.5) = 1.023$

(6) Solve the differential equation $y' + y + xy^2 = 0$ with the initial value of y: $y_0 = 1$, $y_1 = 0.9008$, $y_2 = 0.8066$, $y_3 = 0.722$ corresponding to the values of x: $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ by computing the value of y corresponding to $x = 0.4$. Applying Adams - Bashforth predictor and corrector formula

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Sol^n:

x	y	$y' = -(y + xy^2)$
$x_0 = 0$	$y_0 = 1$	$y'_0 = -(y_0 + x_0 y_0^2) = -(1 + (0)(1)) = -1$
$x_1 = 0.1$	$y_1 = 0.9008$	$y'_1 = -(y_1 + x_1 y_1^2) = -(0.9008 + (0.1)(0.9008)^2) = -0.9819$
$x_2 = 0.2$	$y_2 = 0.8066$	$y'_2 = -(0.8066 + (0.2)(0.8066)^2) = -0.9367$
$x_3 = 0.3$	$y_3 = 0.722$	$y'_3 = -(0.722 + (0.3)(0.722)^2) = -0.8784$
$x_4 = 0.4$	$y_4 = ?$	

we have AB predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0\right]$$

$$y_4^{(p)} = 0.722 + \frac{0.1}{24}\left[55(-0.8784) - 59(-0.9367) + 37(-0.9819) - 9(-1)\right]$$

$$y_4^{(p)} = 0.63709$$

$$y_4' = -(y_4 + x_4 y_4^2)$$

$$= -(0.63709 + (0.4)(0.63709)^2)$$

$$y_4' = -0.79944$$

Next we have $y_4^{(c)} = y_3 + h/24\,(9y_4' + 19y_3' - 5y_2' + y_1')$

$$y_4^{(c)} = 0.722 + \frac{0.1}{24}\left(9(-0.79944) + 19(-0.8784) - 5(-0.9367) + (-0.9819)\right)$$

$$y_4^{(c)} = 0.6379$$

$$y_4' = -(y_4 + x_4 y_4^2)$$

$$= -(0.6379 + (0.4)(0.6379)^2)$$

$$= -0.80066$$

apply corrector formula once again

$$y_4^{(c)} = 0.722 + 0.1/24\left(9(-0.80066) + 19(-0.8784) - 5(-0.9367) + (-0.9819)\right)$$

$$= 0.63785$$

$$y_4^{(c)} = 0.6379$$

Thus $y(0.4) = 0.6379$

(7) Find the value of y at $x=4.4$ by applying Adams - Bashforth method given that $5x\frac{dy}{dx} + y^2 - 2 = 0$ and $y=1$ at $x=4$ initially by generating the other required values from the Taylor's polynomial.

Solⁿ: we need to generate the value of y at $x = 4.1, 4.2, 4.3$

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots$$

Since $x_0 = 4 \quad y_0 = 1$ by data

$$y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2}y''(4) \quad \text{—(1)}$$

Consider $5xy' + y^2 - 2 = 0$ —(2) we obtain

Substituting the initial values

$$(5)(4)y'(4) + 1^2 - 2 = 0$$

$$20y'(4) - 1 = 0$$

$$20y'(4) = 1$$

$$y'(4) = 1/20$$

$$y'(4) = 0.05$$

D. (2) w.r. to x

$$5[xy'' + y'] + 2yy' = 0$$

Substituting the initial values and $y'(4)$

$$5[4y''(4) + y'(4)] + 2y(4)y'(4) = 0$$

$$20y''(4) + 5(0.05) + 2(1)(0.05) = 0$$

$$20y''(4) + 0.25 + 0.1 = 0$$

$$20y''(4) + 0.35 = 0$$

$$20y''(4) = -0.35$$

$$y''(4) = \frac{-0.35}{20}$$

$$y''(4) = -0.0175$$

Since the Value of the second derivative itself is small enough we shall approximate Taylor's Series in (1) upto second degree terms only.

Substitute these value in (1)

$$y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.0175)$$

Now we need to find y at $x = 4.1$, 4.2, 4.3

$$y(4.1) = 1 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.0175)$$

$$y(4.1) = 1.0049$$

$$y(4.2) = 1 + (4.2-4)(0.05) + \frac{(4.2-4)^2}{2}(-0.0175)$$

$$y(4.2) = 1.0097$$

$$y(4.3) = 1 + (4.3-4)(0.05) + \frac{(4.3-4)^2}{2}(-0.0175)$$

$$y(4.3) = 1.0142$$ use these along with $y(4) = 1$ initially

x	y	$y' = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \frac{2-y_0^2}{5x_0} = \frac{2-1^2}{5(4)} = 1/20 = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \frac{2-(1.0049)^2}{5(4.1)} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \frac{2-(1.0097)^2}{5(4.2)} = 0.04669$
$x_3 = 4.3$	$y_3 = 1.0142$	$y_3' = \frac{2-(1.0142)^2}{5(4.3)} = 0.04518$
$x_4 = 4.4$	$y_4 = ?$	

we have predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 1.0142 + \frac{0.1}{24}\left[55(0.04518) - 59(0.04669) + 37(0.0483) - 9(0.05)\right]$$

$$y_4^{(P)} = 1.01864$$

Now, $y_4' = \dfrac{2 - y_4^2}{5x_4}$

$$y_4' = \frac{2 - (1.01864)^2}{5(4.4)}$$

$$y_4' = 0.0437$$

Next we have

$$y_4^{(c)} = y_3 + h/24\,(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$= 1.0142 + \frac{0.1}{24}\left(9(0.0437) + 19(0.04518) - 5(0.04669) + 0.0483\right)$$

$$y_4^{(c)} = 1.0186$$

Now, $y_4' = \dfrac{2 - y_4^2}{5x_4}$

$$y_4' = \frac{2 - (1.0186)^2}{5(4.4)}$$

$$y_4' = 0.0437$$

again put this value in $y_4^{(c)}$

$$\therefore y_4^{(c)} = 1.0142 + \frac{0.1}{24}\left[9(0.0437) + 19(0.04518) - 5(0.04669) + 0.0483\right]$$

$$y_4^{(c)} = 1.0186$$

thus $y(4.4) = 1.0186$

⑧ Use Taylor's Series method (up to 3rd derivative term) to find y at $x=0.1, 0.2, 0.3$ given that $\frac{dy}{dx} = x^2+y^2$ with $y(0)=1$. Apply Milne's predictor corrector formula to find $y(0.4)$ using the generated set of initial values.

Sol^n: By data

$$\frac{dy}{dx} = x^2+y^2, \quad y_0 = 1 \quad x_0 = 0$$

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

$$y(x) = y(0) + (x-0)y'(0) + \frac{(x-0)^2}{2}y''(0) + \frac{(x-0)^3}{6}y'''(0)$$

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \quad \text{—①}$$

Consider $y' = x^2+y^2$ —②

$$y'(0) = 0^2+(y(0))^2$$

$$y'(0) = 0+(1)^2$$

$$y'(0) = 1$$

D.② w.r.to x

$$y'' = 2x + 2yy' \quad \text{—③}$$

$$y''(0) = 2(0) + 2y(0)y'(0)$$

$$y''(0) = 0 + 2(1)(1)$$

$$y''(0) = 2$$

D

y'''

y'''

y'''

y''

D. ③ w.r. to x

$$y''' = 2 + 2[y'y'' + y'y']$$

$$y''' = 2 + 2y'y'' + 2(y')^2$$

$$y'''(0) = 2 + 2y'(0)y''(0) + 2(y'(0))^2$$

$$y'''(0) = 2 + 2(1)(2) + 2(1)^2$$

$$y'''(0) = 2 + 4 + 2$$

$$y'''(0) = 8$$

put all these in ①

$$y(x) = 1 + x(1) + \frac{x^2}{2}(2) + \frac{x^3}{6}(8)$$

Now, we need to find y at $x = 0.1,\ 0.2,\ 0.3$

$$y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{\cancel{2}}(\cancel{2}) + \frac{(0.1)^3}{6} \times 8$$

$$y(0.1) = 1.1113$$

$$y(0.2) = 1 + (0.2)(1) + \frac{(0.2)^2}{\cancel{2}} \times \cancel{2} + \frac{(0.2)^3}{6} \times 8$$

$$y(0.2) = 1.2507$$

$$y(0.3) = 1 + (0.3)(1) + \frac{(0.3)^2}{2} \times 2 + \frac{(0.3)^3}{6} \times 8$$

$$y(0.3) = 1.426$$

Using these values along with $y(0) = 1$ initially, we prepare the following table

P.T.O

x	y	$y' = x^2 + y^2$
$x_0 = 0$	$y_0 = 1$	$y_0' = 0 + 1 = 1$
$x_1 = 0.1$	$y_1 = 1.1113$	$y_1' = (0.1)^2 + (1.1113)^2 = 1.2449$
$x_2 = 0.2$	$y_2 = 1.2507$	$y_2' = (0.2)^2 + (1.2507)^2 = 1.60425$
$x_3 = 0.3$	$y_3 = 1.426$	$y_3' = (0.3)^2 + (1.426)^2 = 2.12347$
$x_4 = 0.4$	$y_4 = ?$	

Consider $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.2449) - (1.60425) + 2(2.12347)\right]$$

$$y_4^{(P)} = 1.68433$$

hence $y_4' = x_4^2 + y_4^2$

$$y_4' = (0.4)^2 + (1.68433)^2$$

$$y_4' = 2.9969$$

Next we have, $y_4^{(C)} = y_2 + h/3\,(y_2' + 4y_3' + y_4')$

$$y_4^{(C)} = 1.2507 + \frac{0.1}{3}\left[1.60425 + 4(2.12347) + 2.9969\right]$$

$$= 1.6872$$

$$y_4' = x_4^2 + y_4^2$$

$$y_4' = (0.4)^2 + (1.6872)^2$$

$$y_4' = 3.0066$$

put y_4' again in the corrector formula

$$y_4^{(c)} = 1.2507 + \frac{0.1}{3}\left[1.60425 + 4(2.12347) + 3.0066\right]$$

$$y_4^{(c)} = 1.6875$$

$$y_4' = x_4^2 + y_4^2$$

$$y_4' = (0.4)^2 + (1.6875)^2$$

$$y_4' = 3.0076$$

put y_4' again in the corrector formula

$$y_4^{(c)} = 1.2507 + \frac{0.1}{3}\left[1.60425 + 4(2.12347) + 3.0076\right]$$

$$y_4^{(c)} = 1.6875$$

Thus $y(0.4) = 1.6875$

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module-05

Numerical Solution of Second order Ordinary differential equations

Introduction:

The given Second order ODE with two initial Conditions will reduce to two first order simultaneous ODEs which can be solved.

Let $y'' = g(x, y, y')$ with the initial Conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ be the second order DE.

Now, Let $y' = \frac{dy}{dx} = z$.

This gives $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}$

The given second order DE assumes the form: $\frac{dz}{dx} = g(x, y, z)$ with the conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ where y_0' is denoted by z_0.

Hence, we now have two first order simultaneous ODEs.

① $\frac{dy}{dx} = z$ and ② $\frac{dz}{dx} = g(x, y, z)$ with $y(x_0) = y_0$ and $z(x_0) = z_0$

Taking $f(x, y, z) = z$, we now have the following system of equations for solving

$\frac{dy}{dx} = f(x, y, z)$, $\frac{dz}{dx} = g(x, y, z)$;

$y(x_0) = y_0$ and $z(x_0) = z_0$

Runge - kutta Method

we have to compute $y(x_0+h)$ and if required $y'(x_0+h) = z(x_0+h)$.

we need to first compute the following

$k_1 = hf(x_0, y_0, z_0)$; $l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$;

$l_2 = hg(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$;

$l_3 = hg(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$;

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

The required

$y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and $y'(x_0+h) = z(x_0+h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

Problems

P.T.O.

① Given $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, $y(0) = 1$, $y'(0) = 0$. Evaluate $y(0.1)$ using Runge kutta method of order 4.

Sol^{n}: By data

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1, \quad y(0) = 1 \text{ and } y'(0) = 0$$

$y(0.1) = ?$

$y = 1$, $y' = 0$ at $x = 0$

$y' = \frac{dy}{dx} = z$

$y'' = \frac{d^2y}{dx^2} = z'$

putting, $\frac{dy}{dx} = z$

and Differentiating w.r. to x we obtain

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

So that given equation becomes

$$\frac{dz}{dx} - x^2 z - 2xy = 1$$

hence, we have a System of equations

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = 1 + 2xy + x^2 z \quad \text{where}$$

$y(0) = 1$, $y'(0) = 0$, $y'_0 = 0$

$y_0 = 1$, $z_0 = 0$, $x_0 = 0$

Let, $f(x, y, z) = z$, $g(x, y, z) = 1 + 2xy + x^2 z$

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$ ($y'_0 = 0$) and let us take $h = 0.1$

$x = x_0 + h$

$0.1 = 0 + h$

$h = 0.1$

we shall first find the following

$K_1 = hf(x_0, y_0, z_0)$

$K_1 = (0.1) f(0, 1, 0)$

$K_1 = (0.1)(0)$

$\underline{\underline{K_1 = 0}}$

$l_1 = hg(x_0, y_0, z_0)$

$l_1 = (0.1) g(0, 1, 0)$

$l_1 = (0.1)[1 + 2(0)(1) + (0)(0)]$

$l_1 = (0.1)(1)$

$l_1 = 0.1$ //

$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.1) f\left(0 + \frac{0.1}{2}, 1 + 0/2, 0 + \frac{0.1}{2}\right)$

$= (0.1) f(0.05, 1, 0.05)$

$= (0.1)(0.05)$

$\underline{\underline{K_2 = 0.005}}$

$l_2 = hg(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.1) g(0 + 0.1/2, 1 + 0/2, 0 + 0.1/2)$

$= (0.1) g(0.05, 1, 0.05)$

$= (0.1)[1 + 2(0.05)(1) + (0.05)^2(0.05)]$

$\underline{\underline{l_2 = 0.110012}}$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$

$= (0.1) f\left(0 + 0.1/2, 1 + \frac{0.005}{2}, 0 + \frac{0.110012}{2}\right)$

$= (0.1) f(0.05, 1.0025, 0.055)$

$= (0.1)(0.055)$

$= 0.0055$ //

$l_3 = hg(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$

$= (0.1) g\left(0 + 0.1/2, 1 + \frac{0.005}{2}, 0 + \frac{0.110012}{2}\right)$

$= (0.1) g(0.05, 1.0025, 0.055)$

$= (0.1)[1 + 2(0.05)(1.0025) + (0.05)^2(0.055)]$

$= 0.110038$ //

$K_4 = hf(x_0+h,\ y_0+K_3,\ z_0+l_3)$

$= (0.1) f(0+0.1,\ 1+0.0055,\ 0+0.110038)$

$= (0.1) f(0.1,\ 1.0055,\ 0.110038)$

$= (0.1)(0.110038)$

$K_4 = 0.011$

$l_4 = hg(x_0+h,\ y_0+K_3,\ z_0+l_3)$

$= (0.1) g(0+0.1,\ 1+0.0055,\ 0+0.110038)$

$= (0.1) g(0.1,\ 1.0055,\ 0.110038)$

$= (0.1)[1+2(0.1)(1.0055)+(0.1)^2(0.110038)]$

$l_4 = 0.12022$

we have to find $y(0.1)$

$y(x_0+h) = y_0 + \frac{1}{6}(K_1+2K_2+2K_3+K_4)$

$y(0.1) = 1 + \frac{1}{6}(0+2(0.005)+2(0.0055)+0.011)$

$y(0.1) = 1.0053$

NOTE: $Z(x_0+h) = Z_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4)$

$Z(0+0.1) = 0 + \frac{1}{6}(0.1+2(0.11012)+2(0.110038) + 0.12022)$

See In this problem they are not asked to find $Z(0.1)$

$Z(0.1) = 0.110089$

② By Runge-Kutta method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$ for $x = 0.2$ correct to four decimal places, using the initial conditions $y=1$ and $y'=0$ when $x=0$

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Sol: By data $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, $y_0 = 1$, $z_0 = 0$, $x_0 = 0$

$y(0.2) = ?$

put $\frac{dy}{dx} = z$

D. w. r. to x

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Now given equation becomes

$$\frac{dz}{dx} = xz^2 - y^2 \quad \text{with } y_0 = 1,\ z_0 = 0 \text{ at } x_0 = 0$$

$y' = 0$

hence we have

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = xz^2 - y^2$$

Let $f(x, y, z) = z$, $g(x, y, z) = xz^2 - y^2$,

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$

$x_0 + h = 0.2$

$0 + h = 0.2$

$h = 0.2$

we shall find the following

$K_1 = hf(x_0, y_0, z_0)$

$K_1 = (0.2) f(0, 1, 0)$

$= (0.2)(0)$

$K_1 = 0$

$l_1 = hg(x_0, y_0, z_0)$

$= (0.2) g(0, 1, 0)$

$= (0.2)[(0)(0)^2 - 1^2]$

$= (0.2)[-1]$

$l_1 = -0.2$

$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2,\ z_0 + l_1/2)$

$= (0.2) f\left(0 + \frac{0.2}{2},\ 1 + 0/2,\ 0 + \frac{(-0.2)}{2}\right)$

$= (0.2) f(0.1, 1, -0.1)$

$= (0.2)(-0.1)$

$K_2 = -0.02$

$l_2 = hg(x_0 + h/2,\ y_0 + k_1/2,\ z_0 + l_1/2)$

$= (0.2)g\left(0 + \frac{0.2}{2},\ 1 + 0/2,\ 0 + \frac{(-0.2)}{2}\right)$

$= (0.2)g(0.1,\ 1,\ -0.1)$

$= (0.2)\left[(0.1)(-0.1)^2 - (1)^2\right]$

$= (0.2)[-0.999]$

$l_2 = -0.1998$ //

$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + l_2/2)$

$= (0.2)f\left(0 + 0.2/2,\ 1 + \frac{(-0.02)}{2},\ 0 + \frac{(-0.1998)}{2}\right)$

$= (0.2)f(0.1,\ 0.99,\ -0.0999)$

$= (0.2)(-0.0999)$

$k_3 = -0.01998$ //

$l_3 = hg(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + l_2/2)$

$= (0.2)g\left(0 + 0.2/2,\ 1 + \frac{(-0.02)}{2},\ 0 + \frac{(-0.1998)}{2}\right)$

$= (0.2)g(0.1,\ 0.99,\ 0.0999)$

$= (0.2)\left[(0.1)(0.0999)^2 - (0.99)^2\right]$

$= (0.2)[-0.979101999]$

$l_3 = -0.1958$ //

$k_4 = hf(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$

$= (0.2)f(0 + 0.2,\ 1 + (-0.01998),\ 0 + (-0.1958))$

$= (0.2)f(0.2,\ 0.98002,\ -0.1958)$

$= (0.2)(-0.1958)$

$k_4 = -0.03916$

$l_4 = hg(x_0+h,\ y_0+k_3,\ z_0+l_3)$

$= (0.2)\, g(0+0.2,\ 1-0.01998,\ 0-0.1958)$

$= (0.2)\, g(0.2,\ 0.98002,\ -0.1958)$

$= (0.2)\left((0.2)(-0.1958)^2 - (0.98002)^2\right)$

$l_4 = -0.19055$

we need to find $y(0.2)$

$\therefore y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(0.2) = 1 + \frac{1}{6}(0 + 2(-0.02) + 2(-0.01998) - 0.03916)$

$\boxed{y(0.2) = 0.9801}$

NOTE: $Z(x_0+h) = Z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$Z(0.2) = 0 + \frac{1}{6}\left(-0.2 + 2(-0.1998) + 2(-0.1958) + (-0.19055)\right)$

$\boxed{Z(0.2) = -0.1969}$

(3) Compute $y(0.1)$ given $\frac{d^2y}{dx^2} = y^3$ and $y = 10$, $\frac{dy}{dx} = 5$ at $x = 0$ by Runge kutta method of fourth order.

Sol^n: By data $\frac{d^2y}{dx^2} = y^3$, $y_0 = 10$, $\frac{dy}{dx} = y' = 5$ at $x_0 = 0$

$y(0.1) = ?$

put $\frac{dy}{dx} = Z$

D. w. r. to x

$\frac{d^2y}{dx^2} = \frac{dZ}{dx}$

given equation becomes

$\frac{dZ}{dx} = y^3$

hence we have system of equations

$\frac{dy}{dx} = Z$ and $\frac{dZ}{dx} = y^3$

$x_0 + h = 0.1$

$h = 0.1 - x_0$

$h = 0.1 - 0$

$h = 0.1$

Let $f(x, y, z) = z$ and $g(x, y, z) = z^3$

with $x_0 = 0$, $y_0 = 10$, $z_0 = 5$

we shall first find the following

$K_1 = hf(x_0, y_0, z_0)$

$K_1 = (0.1) f(0, 10, 5)$

$= (0.1)(5)$

$K_1 = (0.5)$

$l_1 = hg(x_0, y_0, z_0)$

$= (0.1) g(0, 10, 5)$

$= (0.1)(10)^3$

$= 100$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.1) f\left(0 + \frac{0.1}{2}, 10 + \frac{0.5}{2}, 5 + \frac{100}{2}\right)$

$= (0.1) f(0.05, 10.25, 55)$

$= (0.1)(55)$

$= 5.5$

$l_2 = hg(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.1) g(0.05, 10.25, 55)$

$= (0.1)(10.25)^3$

$= 107.68$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$

$= (0.1) f\left(0 + 0.1/2, 10 + 5.5/2, 5 + \frac{107.68}{2}\right)$

$= (0.1) f(0.05, 12.75, 58.84)$

$= (0.1)(58.84)$

$= 5.884$

$l_3 = hg(x_0 + h/2, y_0 + K_2/2, z_0 + l_2/2)$

$= (0.1) g(0.05, 12.75, 58.84)$

$= (0.1)(12.75)^3$

$= 207.267$

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$K_4 = hf(x_0+h,\ y_0+K_3,\ z_0+l_3)$

$= (0.1)f(0+0.1,\ 10+5.884,\ 5+207.267)$

$= (0.1)f(0.1,\ 15.884,\ 212.267)$

$= (0.1)(212.267)$

$= 21.2267$

$l_4 = hg(x_0+h,\ y_0+K_3,\ z_0+l_3)$

$= (0.1)g(0.1,\ 15.884,\ 212.267)$

$= (0.1)(15.884)^3$

$= 400.75$

we have to find $y(0.1)$

$y(x_0+h) = y_0 + 1/6\,(K_1 + 2K_2 + 2K_3 + K_4)$

$= 10 + 1/6\,(0.5 + 2(5.5) + 2(5.884) + 21.2267)$

$y(0.1) = 17.4157$

(4) June 2018 Given $y'' - xy' - y = 0$ with the initial conditions $y(0)=1$, $y'(0)=0$, compute $y(0.2)$ and $y'(0.2)$ using fourth order Runge Kutta method.

Soln: By data $y'' - xy' - y = 0$

$y(0) = 1,\ y'(0) = 0$

$y(0.2) = ?$ and $y'(0.2) = ?$

put $\frac{dy}{dx} = z$

D.w.r. to x

$\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Given equation becomes

$y'' - xy' - y = 0$

$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$

$\frac{dz}{dx} - xz - y = 0$

$\frac{dz}{dx} = y + xz$

Now, we have system of equations

$\frac{dy}{dx} = z$ and $\frac{dz}{dx} = y + xz$

with $x_0 = 0, y_0 = 1, z_0 = 0$ $\quad y'_0 = 0$

Let $f(x, y, z) = z$ and $g(x, y, z) = y + xz$

with $x_0 = 0, y_0 = 1, z_0 = 0$ $\qquad x_0 + h = 0.2$

we shall find the following $\qquad h = 0.2 - x_0$

$\qquad h = 0.2 - 0$

$K_1 = hf(x_0, y_0, z_0)$ $\qquad h = 0.2$

$K_1 = 0.2 f(0, 1, 0)$

$= 0.2(0)$

$K_1 = 0$

$l_1 = hg(x_0, y_0, z_0)$

$l_1 = (0.2) g(0, 1, 0)$

$= (0.2)[1 + (0)(0)]$

$= (0.2)(1)$

$= 0.2$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + 0/2, 0 + \frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1, 0.1)$

$= (0.2)(0.1)$

$= 0.02$

$l_2 = hg(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$= (0.2) g(0.1, 1, 0.1)$

$= (0.2)[1 + (0.1)(0.1)]$

$= 0.202$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= (0.2) f(0 + \frac{0.2}{2}, 1 + \frac{0.02}{2}, 0 + \frac{0.202}{2})$

$= (0.2) f(0.1, 1.01, 0.101)$

$= (0.2)(0.101)$

$k_3 = 0.0202$

$l_3 = hg(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= (0.2) g(0.1, 1.01, 0.101)$

$= (0.2)[1.01 + (0.1)(0.101)]$

$l_3 = 0.20402$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.2) f(0 + 0.2, 1 + 0.0202, 0 + 0.20402)$

$= (0.2) f(0.2, 1.0202, 0.20402)$

$= (0.2)(0.20402)$

$= 0.0408$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.2) g(0.2, 1.0202, 0.20402)$

$= (0.2)[1.0202 + (0.2)(0.20402)]$

$l_4 = 0.2122$

Now we have to find $y(0.2)$ and $y'(0.2)$ (or) $z(0.2)$

we have

$$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$y(0.2) = 1 + \frac{1}{6}(0 + 2(0.02) + 2(0.0202) + 0.0408)$$

$$y(0.2) = 1.0202$$

Then we have to find $y'(0.2)$ or $z(0.2)$

$$z(x_0+h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 0 + \frac{1}{6}(0.2 + 2(0.202) + 2(0.20402) + 0.21$$

$$z(0.2) = 0.20404 \text{ (or) } y'(0.2) = 0.20404$$

Thus $y(0.2) = 1.0202$ and $y'(0.2) = 0.20404$

(5) Obtain the value of x and $\frac{dx}{dt}$ when $t=0.1$ given that x satisfies the equation $\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$ and $x=3$, $\frac{dx}{dt}=0$ when $t=0$ initially. Use 4th order Runge Kutta method.

No Need

Soln: By data

$$\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x, \quad x = 3, \frac{dx}{dt} = 0, t = 0$$

put $\frac{dx}{dt} = y$

D.w.r. to t

$$\frac{d^2x}{dt^2} = \frac{dy}{dt}$$

given equation becomes

$\frac{dy}{dt} = ty - 4x$

hence we have system of equations

$\frac{dx}{dt} = y$ and $\frac{dy}{dt} = ty - 4x$

with $t_0 = 0, x_0 = 3, \frac{dx}{dt} = y_0 = 0$

$\cancel{t_0} + h = 0.1$

$h = 0.1 - t_0$

$h = 0.1 - 0$

$h = 0.1$

Let $f(t, x, y) = y$ and $g(t, x, y) = ty - 4x$

with $t_0 = 0, x_0 = 3, y_0 = 0$

we shall find the following

$k_1 = hf(t_0, x_0, y_0)$
$= (0.1) f(0, 3, 0)$
$= (0.1)(0)$
$= 0$

$l_1 = hg(t_0, x_0, y_0)$
$= (0.1) g(0, 3, 0)$
$= (0.1)[(0)(0) - 4(3)]$
$l_1 = -1.2$

$k_2 = hf(t_0 + h/2, x_0 + k_1/2, y_0 + l_1/2)$
$= (0.1) f(0.05, 3, -0.6)$
$= (0.1)(-0.6)$
$= -0.06$

$l_2 = hg(t_0 + h/2, x_0 + k_1/2, y_0 + l_1/2)$
$= (0.1) g(0.05, 3, -0.6)$
$= (0.1)[(0.05)(-0.6) - 12]$
$= -1.203$

$k_3 = hf(t_0 + h/2, x_0 + k_2/2, y_0 + l_2/2)$
$= (0.1) f(0.05, 2.97, -0.6015)$

$k_3 = (0.1)(-0.6015)$

$\underline{\underline{k_3}} = -0.06015$

$l_3 = (0.1)[(0.05)(-0.6015) - 4 \times 2.97]$

$l_3 = \underline{\underline{-1.191}}$

$k_4 = hf(t_0+h,\ x_0+k_3,\ y_0+l_3)$

$= (0.1) f(0.1,\ 2.93985,\ -1.191)$

$= (0.1)(-1.191)$

$k_4 = \underline{\underline{-0.1191}}$

$l_4 = hg(t_0+h,\ x_0+k_3,\ y_0+l_3)$

$= (0.1) g(0.1,\ 2.93985,\ -1.191)$

$= (0.1)[(0.1)(-1.191) - 4 \times 2.93985]$

$l_4 = \underline{\underline{-1.18785}}$

$x(t_0+h) = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\underline{\underline{x(0.1)}} = 2.9401$

$y(t_0+h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$y(0.1) = \underline{\underline{-1.196}}$

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Milne's Method

Method to solve the ODE $y'' = g(x, y, y')$ given a set of four initial values for y and y'.

① consider $y'' = g(x, y, y')$ with initial condition $y(x_0) = y_0$ and $y'(x_0) = y_0'$

② put $y' = \frac{dy}{dx} = z$

D. w. r. to x

$$y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$$

$$\therefore y'' = z'$$

the given differential equation becomes

$z' = g(x, y, z)$ with initial condition $y(x_0) = y_0$ and $z(x_0) = z_0$

③ The given data's are set of x values i.e x_0, x_1, x_2, x_3, x_4

set of y values y_0, y_1, y_2, y_3,

set of y' or z values z_0, z_1, z_2, z_3

④ First apply predictor formula to find $y_4^{(P)}$ and $z_4^{(P)}$

where
$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

⑤ we compute $z_4' = g(x_4, y_4, z_4)$ and then apply corrector formula

where
$$y_4^{(C)} = y_2 + h/3\,(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + h/3\,(z_2' + 4z_3' + z_4')$$

⑥ For better accuracy apply corrector formula repeatedly

Problems

① Apply milne's method to solve $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ given the following table of initial values compute $y(0.4)$

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x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.399
y'	1	1.2103	1.4427	1.699

Sol^n: By data $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$

put $y' = \frac{dy}{dx} = z$

we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$ $\therefore y'' = z'$

Given equation becomes

$$\frac{dz}{dx} = 1 + z$$

(or) $z' = 1 + z$

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 1.1103$	$y_2 = 1.2427$	$y_3 = 1.399$
$y' = z$	$z_0 = 1$	$z_1 = 1.2103$	$z_2 = 1.4427$	$z_3 = 1.699$
$y'' = z' = 1 + z$	$z_0' = 1 + z_0$ $z_0' = 1 + 1 = 2$	$z_1' = 1 + z_1$ $= 1 + 1.2103$ $= 2.2103$	$z_2' = 1 + z_2$ $= 1 + 1.2427$ $= 2.4427$	$z_3' = 1 + z_3$ $= 1 + 1.399$ $= 2.699$

we first consider milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.2103) - 1.4427 + 2(1.699)\right]$$

$$y_4^{(P)} = 1.58345$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left(2z_1' - z_2' + 2z_3'\right)$$

$$z_4^{(P)} = 1 + \frac{4(0.1)}{3}\left(2(2.2103) - (2.4427) + 2(2.699)\right)$$

$$z_4^{(P)} = 1.98345$$

$$z_4' = 1 + z_4$$

$$z_4' = 1 + 1.98345 \Rightarrow z_4' = 2.98345$$

Now, we consider milne's corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(z_2 + 4z_3 + z_4\right)$$

$$= 1.2427 + \frac{0.1}{3}\left(1.4427 + 4(1.699) + 1.98345\right)$$

$$= 1.58344$$

$$z_4^{(c)} = z_2 + \frac{h}{3}\left(z_2' + 4z_3' + z_4'\right)$$

$$= 1.4427 + \frac{0.1}{3}\left(2.4427 + 4(2.699) + 2.98345\right)$$

$$= 1.98344$$

Substituting again in corrector formula

$$y_4^{(c)} = 1.2427 + \frac{0.1}{3}\left[1.4427 + 4(1.699) + 1.98344\right]$$

$$y_4^{(c)} = 1.58344$$

Thus $y(0.4) = 1.58344$

$= \cancel{1.4427 + 0.1/3\,(2.4427}$

$z_4^{(c)} \cancel{= 1.98344}$

∴ $y(0.4) = 1.58344$ and $z(0.4) = 1.98344$

(2) Apply milne's method to compute $y(0.8)$ given that $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ and the following table of initial values.

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

Apply corrector formula twice in preventing the value of y at $x = 0.8$

Sol^{n}: Given $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$

put $y' = \frac{dy}{dx} = z$

D. w. r. to x

we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$ ∴ $y'' = z'$

given equation becomes

$$z' = 1 - 2yz$$

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.02$	$y_2 = 0.0795$	$y_3 = 0.1762$
$y' = z$	$z_0 = 0$	$z_1 = 0.1996$	$z_2 = 0.3937$	$z_3 = 0.5689$
$y'' = z'$ $= 1-2yz$	$z_0' = 1 - 2y_0z_0$ $= 1 - 2(0)(0)$ $z_0' = 1$	$z_1' = 1 - 2y_1z_1$ $z_1' = 0.992$	$z_2' = 1 - 2y_2z_2$ $z_2' = 0.9374$	$z_3' = 1 - 2y_3z_3$ $z_3' = 0.7995$

we first consider milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$= 0 + \frac{4(0.2)}{3}\left(2(0.1996) - (0.3937) + 2(0.5689)\right)$$

$$y_4^{(P)} = 0.30488$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

$$= 0 + \frac{4(0.2)}{3}(2(0.992) - (0.9374) + 2(0.7995))$$

$$z_4^{(P)} = \underline{\underline{0.70549}}$$

$$z_4' = 1 - 2y_4 z_4$$

$$= 1 - 2(0.30488)(0.70549)$$

$$= \underline{\underline{0.56982}}$$

Now we consider milne's corrector formula

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$= 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.70549)$$

$$= \underline{\underline{0.30448}}$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

$$= 0.3937 + \frac{0.2}{3}(0.9374 + 4(0.7995) + 0.56982)$$

$$= \underline{\underline{0.70738}}$$

Substituting the appropriate value in corrector formula

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.70738)$$

$$= \underline{\underline{0.3046}}$$

Thus $\underline{\underline{y(0.8) = 0.3046}}$

③ Obtain the Solution of the equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying milne's method using the following data.

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

Solⁿ: By data $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$

Divide both sides by 2

$$\frac{\cancel{2}}{\cancel{2}}\frac{d^2y}{dx^2} = \frac{4x + \frac{dy}{dx}}{2}$$

$$\frac{d^2y}{dx^2} = \frac{4x}{2} + \frac{1}{2}\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2x + \frac{1}{2}\frac{dy}{dx} \quad \text{—— ①}$$

put $y' = \frac{dy}{dx} = Z$

D. w. r. to x

$$y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = Z' \Rightarrow y'' = Z'$$

Now eqⁿ ① becomes

$$Z' = 2x + \frac{1}{2}Z$$

$$Z' = 2x + \frac{Z}{2}$$

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x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = Z$	$Z_0 = 2$	$Z_1 = 2.3178$	$Z_2 = 2.6725$	$Z_3 = 3.0657$
$y'' = Z' = 2x + Z/2$	$Z_0' = 2x_0 + Z_0/2$ $Z_0' = 3$	$Z_1' = 2x_1 + Z_1/2$ $Z_1' = 3.3589$	$Z_2' = 2x_2 + Z_2/2$ $Z_2' = 3.73625$	$Z_3' = 2x_3 + \frac{Z_3}{2}$ $Z_3' = 4.13285$

we first consider milne's predictor formula

$$y_4^P = y_0 + \frac{4h}{3}(2Z_1 - Z_2 + 2Z_3)$$

$$= 2 + \frac{4(0.1)}{3}\left(2(2.3178) - 2.6725 + 2(3.0657)\right)$$

$$= 3.07926$$

$$y_4^{(P)} = 3.0793$$

$$Z_4^{(P)} = Z_0 + \frac{4h}{3}(2Z_1' - Z_2' + 2Z_3')$$

$$= 2 + \frac{4(0.1)}{3}\left(2(3.3589) - (3.73625) + 2(4.13285)\right)$$

$$Z_4^{(P)} = 3.4996$$

$$Z_4' = 2x_4 + Z_4/2$$

$$= 2(1.4) + \frac{3.4996}{2}$$

$$Z_4' = 4.5498$$

Now, we consider milne's corrector formula

$$y_4^{(c)} = y_2 + h/3\,(Z_2 + 4Z_3 + Z_4)$$

$$= 2.4649 + \frac{0.1}{3}\left(2.6725 + 4(3.0657) + 3.4996\right)$$

$$= 3.07939$$

$y_4^{(c)} = 3.0794$

$$z_4^{(c)} = z_2 + h/3 \left(z_2' + 4z_3' + z_4'\right)$$

$$= 2.6725 + \frac{0.1}{3}\left(3.73625 + 4(4.13285) + 4.5498\right)$$

$z_4^{(c)} = 3.4997$

onygain apply corrector formula using appropriate values

$$y_4^{(c)} = y_2 + h/3 \left(z_2 + 4z_3 + z_4\right)$$

$$= 2.4649 + \frac{0.1}{3}\left(2.6725 + 4(3.0657) + 3.4997\right)$$

$y_4^{(c)} = 3.0794$

④ Given the ODE $y'' + xy' + y = 0$ and the following table of initial values, compute $y(0.4)$ by applying milne's method.

do yourself

x	0	0.1	0.2	0.3
y	1	0.995	0.9801	0.956
y'	0	-0.0995	-0.196	-0.2867

Solⁿ: By data $y'' + xy' + y = 0$

$$y'' = -xy' - y$$

put $y' = \frac{dy}{dx} = z$

we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$

$$\therefore y'' = z'$$

given equation becomes

$z' = -xz - y$

$z' = -(xz + y)$

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 0.995$	$y_2 = 0.9801$	$y_3 = 0.956$
$y' = z$	$z_0 = 0$	$z_1 = -0.0995$	$z_2 = -0.196$	$z_3 = -0.2867$
$y'' = z' = -(xz+y)$	$z_0' = -1$	$z_1' = -0.985$	$z_2' = -0.941$	$z_3' = -0.87$

we first consider milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$= 1 + \frac{4(0.1)}{3}(2(-0.0995) - (-0.196) + 2(-0.2867))$$

$$= \underline{\underline{0.9231}}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

$$= 0 + \frac{4(0.1)}{3}(2(-0.985) - (-0.941) + 2(-0.87))$$

$$= \underline{\underline{-0.3692}}$$

$$z_4' = -(x_4 z_4 + y_4)$$

$$= -(0.4 \times -0.3692 + 0.9231)$$

$$\underline{\underline{z_4' = -0.7754}}$$

Next we have milne's corrector formula

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$= 0.980 + \frac{0.1}{3}(-0.196 + 4(-0.2867) + (-0.3692))$$

$$y_4^{(C)} = 0.9229$$

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$y_4^{(c)} = 0.9229$

$z_4^{(c)} = z_2 + h/3\,(z_2' + 4z_3' + z_4')$

$= -0.196 + \frac{0.1}{3}\left[-0.941 + 4(-0.87) + (-0.7751)\right]$

$z_4^{(c)} = -0.3692$

Once again put this in corrector formula

$y_4^{(c)} = 0.980 + \frac{0.1}{3}\left[-0.196 + 4(-0.2867) - 0.3692\right]$

$= 0.9229$

Thus $y(0.4) = 0.9229$

(5) Applying milne's predictor and corrector formula compute $y(0.8)$ given that y satisfies the equation $y'' = 2yy'$ and y and y' are governed by the following values.

Do yourself

$y(0) = 0$, $y(0.2) = 0.2027$ $y(0.4) = 0.4228$

$y(0.6) = 0.6841$

$y'(0) = 1$, $y'(0.2) = 1.041$, $y'(0.4) = 1.179$

$y'(0.6) = 1.468$

Apply corrector formula twice

Soln: Given $y'' = 2yy'$ — (1)

put $y' = z$

D. w. r. to x

$y'' = z'$

$z' = 2yz$

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.2027$	$y_2 = 0.4228$	$y_3 = 0.6841$
$y' = z$	$z_0 = 1$	$z_1 = 1.041$	$z_2 = 1.179$	$z_3 = 1.468$
$y'' = z' = 2yz$	$z_0' = 0$	$z_1' = 0.422$	$z_2' = 0.997$	$z_3' = 2.009$

milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$y_4^{(P)} = \underline{1.0237}$$

$$z_4^{(P)} = z_0 + 4h/3\,(2z_1' - z_2' + 2z_3')$$

$$= \underline{2.0307}$$

$$z_4' = 2y_4 z_4$$

$$z_4' = \underline{4.1577}$$

corrector formula

$$y_4^{(C)} = y_2 + h/3\,(z_2 + 4z_3 + z_4)$$

$$= \underline{1.0282}$$

$$z_4^{(C)} = z_2 + h/3\,(z_2' + 4z_3' + z_4')$$

$$= \underline{2.0584}$$

applying corrector formula we have

$$y_4^{(C)} = 1.03009$$

Thus $y(0.8) = \underline{1.0301}$

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Continuation

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Variation of a function

Let us consider a function of x, y, y'.
i.e $f(x, y, y') = f(x, y(x), y'(x))$

Suppose we give small increments to y and y' so that they become respectively, $y + h\alpha(x)$, $y' + h\alpha'(x)$
h is small parameter independent of x. Now we have

$$f(x, y+h\alpha(x), y'+h\alpha'(x)) = f(x, y, y') + \left(h\alpha\frac{\partial}{\partial y} + h\alpha'\frac{\partial}{\partial y'}\right)f + \frac{1}{2!}\left(h\alpha\frac{\partial}{\partial y} + h\alpha'\frac{\partial}{\partial y'}\right)^2 f + \cdots$$

by using Taylor's expansion.
[y and y' are treated as variables since x is fixed]
Neglecting second and higher degree terms since h is small parameter, we have

$$f(x, y+h\alpha(x), y'+h\alpha'(x)) - f(x, y, y')$$
$$= h\alpha\frac{\partial f}{\partial y} + h\alpha'\frac{\partial f}{\partial y'}$$

Denoting the LHS of this equation by δf we have

$$\delta f = h\alpha\frac{\partial f}{\partial y} + h\alpha'\frac{\partial f}{\partial y'} \quad \text{—— (1)}$$

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δf is called Variation of f

we have from ①

$$\delta y = h\alpha \frac{\partial y}{\partial y} + h\alpha' \frac{\partial y}{\partial y'}$$

$$\delta y = h\alpha + h\alpha' \cdot (0)$$

$$\delta y = h\alpha$$

$$\therefore \boxed{h\alpha = \delta y} \quad \text{——— (2)}$$

$$\delta y' = h\alpha \frac{\partial y'}{\partial y} + h\alpha' \frac{\partial y'}{\partial y'}$$

$$\delta y' = h\alpha \cdot 0 + h\alpha' (1)$$

$$\delta y' = h\alpha'$$

$$\therefore \boxed{h\alpha' = \delta y'} \quad \text{——— (3)}$$

Using ② and ③ in ① we have

$$\delta f = \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \quad \text{——— (4)}$$

no need

NOTE: Geometrically $y(x)$ and $y(x) + h\alpha(x)$ represents two neighbouring curves. Variation in f represents the change in f from curve to curve.

We now proceed to establish two important properties connected with variational operator δ, differential operator d/dx & integral $\int$

Property - I

$$\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$$

proof: $\delta\left(\frac{dy}{dx}\right) = \delta y' = h\alpha'$ by using (3)

$= h\frac{d\alpha}{dx}$

$= \frac{d(h\alpha)}{dx}$ since h is independent of x

$= \frac{d}{dx}(\delta y)$ by using (2)

$$\therefore \boxed{\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)}$$

Functionals

Let S be a set of functions of a single variable x defined over an interval (x_1, x_2)
Then any function which assigns to each function in S a unique real value is called a functional.
In other words functional is a mapping from functions to real numbers.
Consider a function of the form $f(x, y, y')$ where y' is derivative of y w.r.to x and $x \in (x_1, x_2)$

The integral $I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$ is a functional [a standard form]

It can be easily seen that for every $y(x)$, $I(y)$ give a real value.

Examply of functionals

① $\int_0^1 x + (y')^2\, dx$ ② $\int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$

Property - 2

If $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ then

$$\delta \int_{x_1}^{x_2} f(x, y, y')\, dx = \int_{x_1}^{x_2} \delta f(x, y, y')\, dx$$

i.e to say that the variational of a functional associated with $f(x, y, y')$ is equal to the functional associated with variation of f.

Proof: $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ is functional

Since the value of I depends on y and y' we have by using the result connected with variation

$$\delta f = \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y'$$

$$\therefore \delta I = \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial y'}\delta y'$$

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$$\delta I = \left\{\int_{x_1}^{x_2}\frac{\partial}{\partial y}\left[f(x,y,y')\right]dx\right\}\delta y + \left\{\int_{x_1}^{x_2}\frac{\partial}{\partial y'}\left[f(x,y,y')\right]dx\right\}\delta y'$$

i.e. $$\delta I = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\delta y\,dx + \int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\delta y'\,dx$$

$$\delta I = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right]dx$$

$$\delta I = \int_{x_1}^{x_2}\delta f\,dx$$

Thus $$\delta\int_{x_1}^{x_2} f(x,y,y')\,dx = \int_{x_1}^{x_2}\delta f(x,y,y')\,dx$$

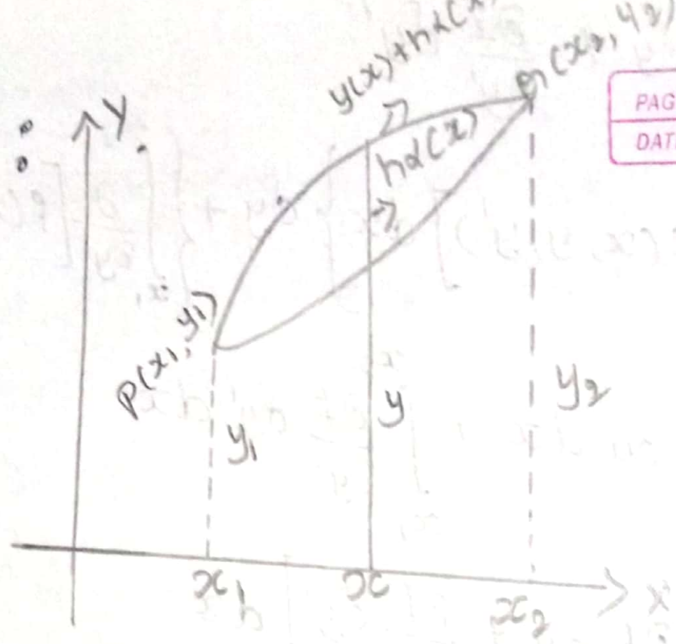
[Dec 17, 18]

Euler's Equation

Statement: A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x,y,y')\,dx$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ to be an extremum is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \text{[Euler's equation]}$$

Proof:



Let I be an extremum along some curve $y = y(x)$ passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$

Also, let $y = y(x) + h\alpha(x)$ —— (1)

be the neighbouring curve (where h is small) joining these points so that we must have $\alpha(x_1) = 0$ at P and $\alpha(x_2) = 0$ at Q —— (2)

when $h = 0$ these two curves coincide thus making I an extremum.

when $h = 0$ these two curves coincide thus making I an extremum.

i.e to say that

$$I = \int_{x_1}^{x_2} f\left(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x)\right) dx$$

Applying chain rule for the partial derivative in RHS, we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right] dx \quad \text{—— (3)}$$

But h is independent of x and hence $\frac{\partial x}{\partial h} = 0$

Let us consider (1) and D.w.r. to x

$\therefore \; y' = y'(x) + h\alpha'(x)$ —— (4)

Also, we have from (1), $\frac{\partial y}{\partial h} = \alpha(x)$ and

from (4) $\frac{\partial y'}{\partial h} = \alpha'(x)$

using these results in (3) we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x) \right] dx \quad \text{—— (5)}$$

Keeping the first term in the RHS of (5) as it is and integrating the second term by parts we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x)\,dx + \left\{ \left[\frac{\partial f}{\partial y'}\alpha(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx \right\}$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x)\,dx + \left\{ \frac{\partial f}{\partial y'}\alpha(x_2) - \frac{\partial f}{\partial y'}\alpha(x_1) \right\}$$
$$- \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

But $\alpha(x_1) = 0 = \alpha(x_2)$ and we have by combining from (2) the two integrals.

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \alpha(x)\,dx$$

But we have already stated that $\frac{dI}{dh}$ must be zero when $h=0$ for I to be an extremum.

hence integrand in the RHS must be zero

Since $\alpha(x)$ is arbitary we must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

This is the required Euler's equation being the necessary condition for the extremum of functional $I = \int_{x_1}^{x_2} f(x, y, y')$

Theorem: The necessary condition for the functional $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ to be an extremum is $\delta I = 0$

Proof: Retrace the steps as in the derivation of Euler's equation upto the stage of arriving at equation (5)

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x)\right] dx \quad \text{--- (5)}$$

we have $\delta I = \delta \int_{x_1}^{x_2} f(x, y, y')\, dx$

Since δ and $\int$ are commutative with each other we have

$$\delta I = \int_{x_1}^{x_2} \delta f(x, y, y')\, dx$$

using the expression for the variation of f being δf in the RHS, we have

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right] dx$$

But $\delta y = h\alpha(x)$ and $\delta y' = h\alpha'(x)$

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} h\alpha(x) + \frac{\partial f}{\partial y'} h\alpha'(x) \right] dx$$

$$\delta I = h \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \alpha'(x) \right] dx$$

i.e $\delta I = h \frac{dI}{dh}$, by equation (5)

But $\frac{dI}{dh} = 0$ when $h=0$ is a necessary condition for I to be an extremum.

Thus $\delta I = 0$ also represents the necessary condition for the functional I to be an extremum.

Problems :-

(1) Find the extremal of the functional

$$\int_{x_1}^{x_2} (y' + x^2 y'^2)\, dx$$

(07)

Solve the Euler's equation for the functional $\int_{x_1}^{x_2} (1 + x^2 y') y'\, dx$ [June 2017, 18]

Soln: Let, $f(x, y, y') = y' + x^2 y'^2$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$$0-\frac{d}{dx}\left(1+x^2 2y'\right)=0$$

$$\frac{d}{dx}\left(1+2x^2 y'\right)=0$$

Integrating w.r. to x we get

$1+2x^2y' = K_1$, where K_1 is constant

$$2x^2y' = K_1 - 1$$

$$y' = \frac{K_1-1}{2x^2}$$

(or) $$\frac{dy}{dx} = \frac{K_1-1}{2x^2}$$

on integration w.r. to x

$$y = \frac{K_1-1}{2}\int \frac{1}{x^2}\,dx + C_2$$

$$y = \frac{K_1-1}{2}\times\frac{-1}{x} + C_2$$

$$y = \frac{1-K_1}{2x} + C_2$$

$y = \frac{C_1}{x} + C_2$ where $C_1 = \frac{1-K_1}{2}$

(2) Find the function y which makes the integral $\int_{x_1}^{x_2}(1+xy'+xy'^2)\,dx$ an extremum.

Solⁿ: Let $f(x, y, y') = 1 + xy' + xy'^2$

Euler's equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \text{ becomes}$$

$$0 - \frac{d}{dx}\left(x + 2xy'\right) = 0$$

$$\frac{d}{dx}\left(x + 2xy'\right) = 0$$

on integration w.r. to x

$x + 2xy' = K_1$, where K_1 is constant

$$2xy' = K_1 - x$$

$$y' = \frac{K_1 - x}{2x}$$

or $$\frac{dy}{dx} = \frac{K_1 - x}{2x}$$

$$\frac{dy}{dx} = \frac{K_1}{2x} - \frac{\cancel{x}}{2\cancel{x}}$$

$$\frac{dy}{dx} = \frac{K_1}{2x} - \frac{1}{2}$$

on integration

$$y = \int\left(\frac{K_1}{2x} - \frac{1}{2}\right)dx + C_2$$

$$y = \frac{K_1}{2}\int\frac{1}{x}\,dx - \frac{1}{2}\int 1.\,dx + C_2$$

$$y = \frac{K_1}{2}\log x - \frac{1}{2}x + C_2$$

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$y = c_1 \log x - \frac{x}{2} + c_2$ where $c_1 = K_1/2$

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③ Find the extremal of the functional

$$\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)\, dx$$

Sol^n: Let, $f(x, y, y') = y^2 + y'^2 + 2ye^x$

Euler's equation,

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$$(2y + 2e^x) - \frac{d}{dx}(2y') = 0$$

$$\frac{d}{dx}(2y') = 2y + 2e^x$$

$2\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2y + 2e^x$ or $2y'' = 2y + 2e^x$

$2 \cdot \frac{d^2y}{dx^2} = 2y + 2e^x$ or $y'' = y + e^x$

÷ Both sides by 2

$$\frac{d^2y}{dx^2} = y + e^x$$

$\frac{d^2y}{dx^2} - y = e^x \Rightarrow D^2y - y = e^x$

$(D^2 - 1)y = e^x$ where $D = \frac{d}{dx}$

A.E is $m^2 - 1 = 0 \therefore m = \pm 1$

hence CF, $y_c = c_1 e^x + c_2 e^{-x}$

$PI = y_p = \frac{\phi(x)}{f(D)}$

$y_p = \frac{e^x}{D^2-1}$

replace D by $-a$ i.e
i.e D by -1

$y_p = \frac{e^x}{(-1)^2-1} = \frac{e^x}{0}$ (Dr = 0)

If Dr is zero diff. Dr and x'y x to Nr

$y_p = \frac{x e^x}{2D}$

$y_p = x/2 \int e^x dx$

$y_p = x/2 \, e^x$

we have $y = y_c + y_p$

$y = c_1 e^x + c_2 e^{-x} + xe^x/2$

(4) Find the curve on which the following functional $\int_0^1 [(y')^2 + 12xy] dx$ with $y(0) = 0$ and $y(1) = 1$ can be determined.

Dec17

Soln: Let $I = \int_0^1 [(y')^2 + 12xy] dx$

$f(x, y, y') = (y')^2 + 12xy$

Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$12x - \frac{d}{dx}(2y') = 0$$ (or) $$12x - \frac{d}{dx}(2y') = 0$$

$$12x - 2y'' = 0$$

$$12x - \frac{d}{dx}\left(2 \cdot \frac{dy}{dx}\right) = 0$$

$$12x - 2 \cdot \frac{d^2y}{dx^2} = 0$$

$$12x = 2\frac{d^2y}{dx^2}$$

$\frac{d^2y}{dx^2} = 6x$ and integrating w.r. to x

we get

$$\frac{dy}{dx} = \frac{6x^2}{2} + c_1$$

$$\frac{dy}{dx} = 3x^2 + c_1$$

Again integrating w.r. to x

$$y = \frac{3x^3}{3} + c_1 x + c_2$$

$$y = x^3 + c_1 x + c_2 \quad \text{— } (*)$$

Using the condition $y=0$ at $x=0$ in $(*)$

$$0 = 0 + 0 + c_2 \Rightarrow c_2 = 0$$

and we $y=1$ at $x=1$

$$1 = 1 + c_1 + 0$$

$$c_1 = 1 - 1 \Rightarrow \underline{\underline{c_1 = 0}}$$

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put c_1 and c_2 in ⊛

$y = x^3$ is the required curve

(5) Solve the Variational problem

$\delta \int_0^1 (12xy + y'^2)\,dx$ under the conditions

Dec 18

$y(0) = 3$ and $y(1) = 6$

Let $I = \int_0^1 (12xy + y'^2)\,dx$, $\delta I = 0$ is equivalent to Euler's eqn.

Solⁿ: $f(x, y, y') = 12xy + (y')^2$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$12x - \frac{d}{dx}(2y') = 0$$

$$12x - 2\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad 12x - 2y'' = 0$$

$$\frac{d^2y}{dx^2} = 6x$$

Integrating w.r. to x we get

$$\frac{dy}{dx} = 6\frac{x^2}{2} + c_1$$

$$\frac{dy}{dx} = 3x^2 + c_1$$

Again integrating w.r. to x

$$y = x^3 + c_1 x + c_2 \quad \text{— ⊛}$$

use the condition $y = 3$ at $x = 0$ in ⊛

$$3 = 0 + 0 + c_2 \Rightarrow c_2 = 3$$

put $x = 1$, $y = 6$ in ⊛

$$6 = 1 + c_1 + 3 \Rightarrow c_1 = 6 - 4 = 2$$

$$c_1 = 2$$

put c_1 and c_2 in (*)

$y = x^3 + 2x + 3$ is the required curve

Do yourself

* Solve the Variational problem:

$\delta\int_0^1 (x+y+y'^2)\,dx = 0$ under the conditions $y(0)=1$ and $y(1)=2$.

Ans: $y = \frac{x^2}{4} + \frac{3}{4}x + 1$ $\quad c_2 = 1$, $c_1 = 3/4$

(6*) S.T the functional $\int_{x_1}^{x_2} (y^2 + x^2 y')\,dx$ assumes extreme values on the Straight line $y = x$

Soln: Let, $f(x,y,y') = y^2 + x^2 y'$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$2y - \frac{d}{dx}(x^2) = 0$$

$$\frac{d}{dx}(x^2) = 2y$$

$$\cancel{2}x = \cancel{2}y$$

$x = y$ or $y = x$ is straight line

7. S.T $\int_{x_1}^{x_2} y^2 y'^2 \, dx$ has an extremum when $y(x)$ is of the form $c_1 \cdot \sqrt{x + c_2}$

Sol: Let $f(x, y, y') = y^2 y'^2$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$2yy'^2 - \frac{d}{dx}(2y^2 y') = 0$$

$$yy'^2 - \frac{d}{dx}(y^2 y') = 0$$

$$yy'^2 - (y^2 y'' + 2yy'y') = 0$$

$$yy'^2 - (y^2 y'' + 2yy'^2) = 0$$

$$-y^2 y'' - yy'^2 = 0$$

$$y^2 y'' + yy'^2 = 0$$

$$y[yy'' + y'^2] = 0 \quad (\div \text{B.S by } y)$$

$$yy'' + y'^2 = 0$$

$$\frac{d}{dx}[yy'] = 0$$

on integration we get

$$yy' = K_1$$

$$y\frac{dy}{dx} = K_1$$

$$y\,dy = K_1\,dx$$

$$\int y\,dy = K_1 \int 1\,dx + K_2$$

$\frac{y^2}{2} = K_1 x + K_2$

$y^2 = 2(K_1 x + K_2)$

$y = \sqrt{2(K_1 x + K_2)}$

$y = \sqrt{2K_1\left(x + K_2/K_1\right)}$

Let us denote $c_1 = \sqrt{2K_1}$ and $c_2 = K_2/K_1$

Thus $y = c_1\sqrt{x + c_2}$

(8) S.T an extremal of $\int_{x_1}^{x_2} \left(\frac{y'}{y}\right)^2 dx$ is expressible in the form $y = ae^{bx}$

Soln: Let $f(x, y, y') = \left(\frac{y'}{y}\right)^2$

$f(x, y, y') = \frac{y'^2}{y^2}$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$-\frac{2}{y^3} y'^2 - \frac{d}{dx}\left(\frac{1}{y^2} \times 2y'\right) = 0$

$-2\left[+\frac{y'^2}{y^3} + \frac{d}{dx}\left(\frac{y'}{y^2}\right)\right] = 0$

÷ B.S by -2

$$\frac{y'^2}{y^3}+\frac{d}{dx}\left(\frac{y'}{y^2}\right)=0$$

$$\frac{y'^2}{y^3}+\frac{y^2y''-2yy'y'}{y^4}=0$$

$$\frac{y'^2}{y^3}+\frac{y^2y''-2yy'^2}{y^4}=0$$

$$\frac{y'^2}{y^3}+\frac{\cancel{y}(yy''-2y'^2)}{y^{\cancel{4}}}=0$$

$$\frac{y'^2}{y^3}+\frac{yy''-2y'^2}{y^3}=0$$

$$\frac{y'^2+yy''-2y'^2}{y^3}=0$$

$$y'^2+yy''-2y'^2=0$$

$$yy''-y'^2=0$$

Now it can be put in the form

$\frac{d}{dx}\left(\frac{y'}{y}\right)=0$ on integration w.r. to x

$$\frac{y'}{y}=C_1$$

again integrate

hence, $\int \frac{y'}{y}\,dx = \int c_1\,dx + c_2$

$$\log_e y = c_1 x + c_2$$

$$y = e^{c_1 x + c_2}$$

$$y = e^{c_1 x} \cdot e^{c_2}$$

$y = ae^{bx}$ where $a = e^{c_2}$ and $b = c_1$

(9) Find the curve on which the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx$ with $y(0) = y(\pi/2) = 0$ can be extremified

Sol^{n}: Let $f(x, y, y') = y'^2 - y^2 + 2xy$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$-2y + 2x - \frac{d}{dx}(2y') = 0$$

$$-y + x - \frac{d}{dx}(y') = 0$$

$$-y + x - y'' = 0$$

$$y'' + y = x$$

$$\frac{d^2y}{dx^2} + y = x \Rightarrow D^2y + y = x$$

$$(D^2+1)y = x$$

where $D = \frac{d}{dx}$

A.E is $m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m = \pm\sqrt{-1}$

$m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$$y_p = \frac{\phi(x)}{f(D)}$$

$$y_p = \frac{x}{D^2+1}$$

$$y_p = \frac{x}{1+D^2}$$

$$\begin{array}{r|l} & x \\ \hline 1+D^2 & x \\ & -x \\ \hline & 0 \end{array}$$

$y_p = x$

$y = y_c + y_p$

$y = c_1 \cos x + c_2 \sin x + x$ — (*)

Use the given conditions $y(0) = y(\pi/2) = 0$

i.e $y(0)=0$ and $y(\pi/2)=0$

put $x=0$ & $y=0$ in (*)

$0 = c_1 \cos(0) + c_2 \sin(0) + 0$

$0 = c_1 \Rightarrow c_1 = 0$

put $y=0$ and $x=\pi/2$ in (*)

$0 = c_1 \cos \pi/2 + c_2 \sin \pi/2 + \pi/2$

$0 = (0)(0) + c_2(1) + \pi/2$

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$C_2 = -\pi/2$

put C_1 and C_2 in ⊛

$y = (0)\cos x + \sin x(-\pi/2) + x$

$y = -\pi/2 \sin x + x$ is the required curve

Geodesics

A geodesic on a surface is a curve along which the distance between any two points of the surface is minimum

Standard variational problems

① P.T the shortest distance between two points in a plane is along the straight line joining them or Prove that geodesics on a plane are straight lines. June17, Dec16, 18

Soln: Let $y = y(x)$ be a curve joining two points ~~in a plane is along the,~~ $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the xoy plane.

W.K.T the arc length between P and Q is given by

$$S = \int_{x_1}^{x_2} \frac{ds}{dx}\, dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = I = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

we need to find the curve $y(x)$ such that I is minimum

Let, $f(x, y, y') = \sqrt{1+(y')^2}$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$$0 - \frac{d}{dx}\left[\frac{1}{\cancel{2}\sqrt{1+(y')^2}} \times \cancel{2}y'\right] = 0$$

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right] = 0 \quad \text{(apply } u/v \text{ rule)}$$

$$\frac{y''\sqrt{1+(y')^2} - y'\frac{1}{\cancel{2}\sqrt{1+(y')^2}} \times \cancel{2}y'y''}{\left(\sqrt{1+(y')^2}\right)^2} = 0$$

$$y''\sqrt{1+(y')^2} - (y')^2 y'' \frac{1}{\sqrt{1+(y')^2}} = 0$$

$$\frac{y''\left(\sqrt{1+(y')^2}\right)^2 - (y')^2 y''}{\sqrt{1+(y')^2}} = 0$$

$$y''\left(\cancel{\sqrt{}}\, 1+(y')^2\right)^2 - y''(y')^2 = 0$$

$$y''(1+(y')^2) - y''(y')^2 = 0$$

$$y'' + \cancel{y''(y')^2} - \cancel{y''(y')^2} = 0$$

$$y'' = 0$$

$$\frac{d^2y}{dx^2} = 0$$

Integrate w.r. to x

$$\frac{dy}{dx} = c_1$$

again integrate w.r. to x

$y = c_1 x + c_2$ which is a straight line

(2) June 2018 Find geodesics on a surface given that arc length on the surface is $S = \int_{x_1}^{x_2} \sqrt{x(1+(y')^2)}\, dx$

Solⁿ: we have $f = \sqrt{x(1+(y')^2)}$

which is independent of y

$\therefore$ Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$0 - \frac{d}{dx}\left(\frac{1}{\cancel{2}\sqrt{x(1+(y')^2)}} \times x \cancel{2} y'\right) = 0$$

$$\frac{d}{dx}\left(\frac{xy'}{\sqrt{x(1+(y')^2)}}\right) = 0$$

Integrate w.r. to x

$$\frac{xy'}{\sqrt{x(1+(y')^2)}} = c$$

$$\frac{xy'}{\sqrt{x}\sqrt{1+(y')^2}} = c$$

$$\left\{ \frac{x}{\sqrt{x}} = \frac{\sqrt{x}\sqrt{x}}{\sqrt{x}} = \sqrt{x} \right.$$

$$\sqrt{x}\,\frac{y'}{\sqrt{1+(y')^2}} = c$$

$$\sqrt{x}\,y' = c\sqrt{1+(y')^2}$$

S. on B. S

$$x(y')^2 = c^2(1+(y')^2)$$

$$x(y')^2 = c^2 + c^2(y')^2$$

$$x(y')^2 - c^2(y')^2 = c^2$$

$$(y')^2(x-c^2) = c^2$$

$$(y')^2 = \frac{c^2}{x-c^2}$$

$$y' = \sqrt{\frac{c^2}{x-c^2}}$$

$$y' = \frac{c}{\sqrt{x-c^2}} \quad \text{or} \quad \frac{dy}{dx} = \frac{c}{\sqrt{x-c^2}}$$

integrate w.r. to x

$$y = c\int \frac{1}{\sqrt{x-c^2}}\,dx + c_1$$

$y = 2c\sqrt{x-c^2} + c_1$

$y - c_1 = 2c\sqrt{x-c^2}$

S. on. B. S

$(y-c_1)^2 = 2^2c^2(x-c^2)$

$(y-c_1)^2 = 4c^2(x-c^2)$ is the required geodesic which is a parabola.

③ P.T catenary is the curve which when rotated about a line generate a surface of minimum area.

Do yourself

Soln: we have the expression for the total surface area given by $\int 2\pi y \, ds$ where the curve is rotated about x-axis.

$$\therefore I = \int_{x_1}^{x_2} 2\pi y \frac{ds}{dx} dx$$

$$= \int_{x_1}^{x_2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_{x_1}^{x_2} 2\pi y \sqrt{1+(y')^2} \, dx$$

Since 2π is constant we can as well take $f(x, y, y') = y\sqrt{1+(y')^2}$ which is independent of x.

$\therefore$ It is convenient to take the Euler's equation in the form.

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

i.e, $$y\sqrt{1+(y')^2} - y'\cdot\frac{y}{\cancel{2}\sqrt{1+(y')^2}}\cancel{2}y' = c$$

$$\frac{y\left(\sqrt{1+(y')^2}\right)^2 - (y')^2 y}{\sqrt{1+(y')^2}} = c$$

$$y(1+(y')^2) - y(y')^2 = c\sqrt{1+(y')^2}$$

$$y + \cancel{y(y')^2} - \cancel{y(y')^2} = c\sqrt{1+(y')^2}$$

$$y = c\sqrt{1+(y')^2}$$

S. on B.S

$$y^2 = c^2\left(\sqrt{1+(y')^2}\right)^2$$

$$y^2 = c^2(1+(y')^2)$$

$$y^2 = c^2 + c^2(y')^2$$

$$c^2(y')^2 = y^2 - c^2 \Rightarrow (y')^2 = \frac{y^2-c^2}{c^2}$$

$$y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\,dx$$

$$\int \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\int dx + k$$

$$\cosh^{-1}(y/c) = \frac{x}{c} + k$$

$$y/c = \cosh(x/c + k)$$

$$y = c\cosh(x/c + k)$$

$$y = c\cosh\left(\frac{x+ck}{c}\right)$$

$$y = c\cosh\left(\frac{x+a}{c}\right)$$ where $a = kc$, this is a

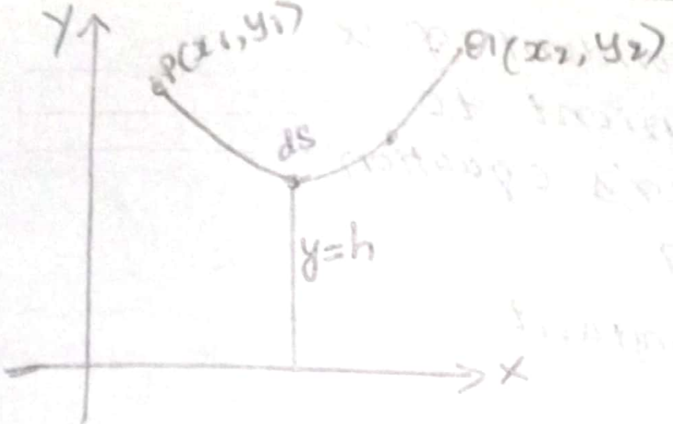
Catenary

* Dec 16, 18

Hanging cable (chain) problem

* A heavy cable hangs freely under gravity between two fixed points S.T the shape of the cable is a catenary.

Solno



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two fixed points of the hanging cable. Let us consider an elementary arc length ds of the cable. Let ρ be the density (mass/unit length) of the cable so that $\rho\, ds$ is the mass of the element. If g is the acceleration due to gravity then the potential energy of the element (m.g.h) is given by $(\rho\, ds) \cdot g \cdot y$ where x-axis is taken as the line of refrence.

$\therefore$ Total potential energy of the cable is given by

$$T = \int_{P}^{Q} (\rho\, ds) \cdot g y \, dx$$

$$= \int_{x_1}^{x_2} \rho g y \frac{ds}{dx} \, dx$$

But $\frac{ds}{dx} = \sqrt{1+(y')^2}$

here, $f(x, y, y') = (\rho g) y \sqrt{1+(y')^2}$

$$= \text{const.}\ y\sqrt{1+(y')^2}$$

which is independent of x

$\therefore$ It is convenient to take the Euler's equation in the form

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

$$y\sqrt{1+(y')^2} - y'\frac{y}{\cancel{2}\sqrt{1+(y')^2}} \times \cancel{2}y' = C$$

$$y\sqrt{1+(y')^2} - \frac{(y')^2 y}{\sqrt{1+(y')^2}} = C$$

$$\frac{y\left(\sqrt{1+(y')^2}\right)^2 - (y')^2 y}{\sqrt{1+(y')^2}} = C$$

$$y(1+(y')^2) - y(y')^2 = C\sqrt{1+(y')^2}$$

$$y + \cancel{y(y')^2} - \cancel{y(y')^2} = C\sqrt{1+(y')^2}$$

$$y = C\sqrt{1+(y')^2}$$

S.on B.S

$$y^2 = c^2(1+(y')^2)$$

$$y^2 = c^2 + c^2(y')^2$$

$$c^2(y')^2 = y^2 - c^2$$

$$(y')^2 = \frac{y^2 - c^2}{c^2}$$

$$y' = \sqrt{\frac{y^2 - c^2}{c^2}}$$

$$y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\,dx$$

$$\int \frac{1}{\sqrt{y^2-c^2}}\,dy = \frac{1}{c}\int dx + K$$

$$\cosh^{-1}(y/c) = x/c + K$$

$$y/c = \cosh(x/c + K)$$

$$y = c\cosh(x/c + K)$$

$$y = c\cosh\left(\frac{x+cK}{c}\right)$$

$$y = c\cosh\left(\frac{x+a}{c}\right) \quad \text{where } a = Kc$$

which is catenary

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